Matita Tutorial

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This tutorial provides a pragmatic introduction to the main functionalities of the Matita interactive theorem prover, offering a guided tour through a set of not so trivial examples in the field of software specification and verification.

Contents

0. GETTING STARTED

Matita [4] is a dependently-typed interactive prover under development at the Computer Science Department of the University of Bologna.

An interactive prover is a software tool aiding the development of formal proofs by man-machine collaboration. It provides a formal language where mathematical definitions, executable algorithms and theorems coexist, and an interactive environment keeping the current status of the proof, and updating it according to commands (usually called tactics) issued by the user [13, 24].

This tutorial provides an introduction to the system, explicitly addressed to absolute beginners, and does not require previous knowledge about interactive theorem proving or type theory. An executable version of the tutorial is available in the `/usr/share/matita/lib/tutorial` directory after having installed Matita (see next Section). The reader is supposed to run the executable tutorial while reading the current document: in this document we only illustrate those code snapshots that showcase noteworthy concepts and techniques for the first time.

The tutorial is also a companion document to the user manual of Matita, that can be browsed from the `Help` menu of the application. The manual provides the comprehensive list of commands of Matita, comprising their syntax and semantics.

0.1 Installing Matita

At present, Matita only works on Linux-based systems. Both Debian and Ubuntu systems have packages called "matita" in the standard system repositories, but we do not suggest to use them, since they would install an out-of-date and incompatible version of the Matita system.

If you are running a Debian-based system with APT installed, you should first of all install the required dependencies by issuing the following command at a terminal window[1]

```
apt-get install ocaml ocaml-findlib libgdome2-ocaml-dev
  liblablgtk2-ocaml-dev liblablgtksourceview-ocaml-dev
  libsqlite3-ocaml-dev libocamlnet-ocaml-dev libzip-ocaml-dev
  libhttp-ocaml-dev ocaml-ulex08 libexpat-ocaml-dev
  libmysql-ocaml-dev camlp5
```

The next step is to prepare a directory for the Matita sources and binaries and enter it; for instance, issue the following series of commands:

```
$ cd ~
$ mkdir Matita
$ cd Matita
```

We shall henceforth refer to this directory as `$MATITA_HOME`. You should now download and unpack from the Matita download page at http://matita.cs.unibo.it/download.shtml the most recent version of the Matita development source tarball; at present this is `matita_130312.tar.gz`:

[1]If you are running the latest Ubuntu release the package `liblablgtksourceview-ocaml-dev` has been superseded by `liblablgtksourceview2-ocaml-dev`

```
$ wget http://matita.cs.unibo.it/sources/matita_130312.tar.gz
$ tar -xzf  matita_130312.tar.gz
```

In `$MATITA_HOME` you should now be left with two further subdirectories, `matita` and `components`, as well as numerous makefiles and auto-configuration scripts. Build the configuration script with the following command:[2]

```
$ autoconf configure.ac > configure
$ chmod +x configure
$ ./configure
```

This will check that all needed tools and libraries are installed and prepare the sources for compilation and installation. Then, type:

```
$ make world
```

All being well, the previous command will build the various Matita-related binaries and their optimised counterparts and place them in `$MATITA_HOME/matita`. In particular, check for the presence of the optimised Matita binary, `matita.opt`, in this subdirectory.

0.2 Preparing a working directory

Before you start editing proof scripts you must prepare a working directory; this can be anywhere in your file systemâĂŹs file hierarchy and does not need to be a subdirectory of `$MATITA_HOME`. For example:

```
$ cd ~
$ mkdir ProofScripts
$ cd ProofScripts
```

We shall refer to this directory as `$SCRIPTS_HOME`, henceforth. In `$SCRIPTS_HOME` create a file called `root` containing the following declaration:

```
baseuri=cic:/matita
```

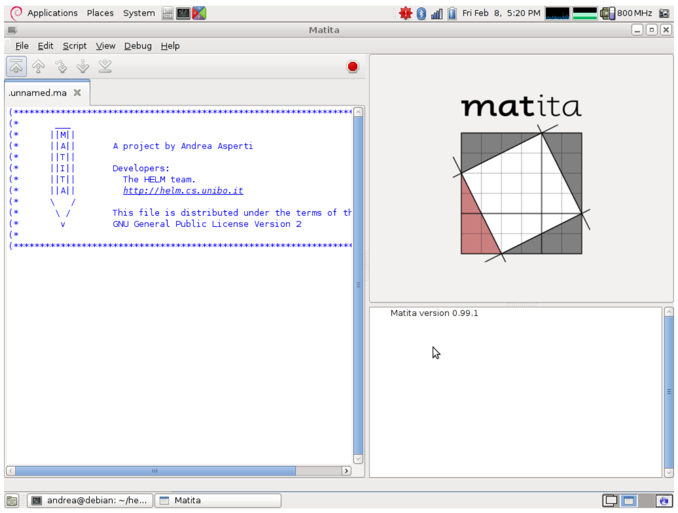
Congratulations, you are ready to start proving things!

0.3 Matita interface

In order to check that everything is up and running, let us perform a simple experiment. Open Matita by invoking `$MATITA_HOME/matita/matita.opt` from a command line. A window should appear on your screen with the shape in Figure 0.3

The interface [8] is divided into three subpanes: one on the left and two stacked vertically on the right. The pane in the top right contains, at the moment, the Matita logo: when you are in the middle of a proof, it will be used to visualize open goals in a sequent like fashion; the pane beneath it is a read only area meant for error and log messages; finally, the pane on the left is an editor pane. When you open a new file, the latter pane contains a default comment with copyright information. Let us observe, by the way, that Matita's style of comments follow the

[2]If `autoconf` is not installed in your system, you will have to install it using the command `apt-get install autoconf` first



```
(****************************************************************
(*
(*      ||M||
(*      ||A||       A project by Andrea Asperti
(*      ||T||
(*      ||I||       Developers:
(*      ||T||         The HELM team.
(*      ||A||         http://helm.cs.unibo.it
(*      \   /
(*       \ /        This file is distributed under the terms of th
(*        v         GNU General Public License Version 2
(*
(****************************************************************
```

```
Matita version 0.99.1
```

Fig. 1. Matita interface

Standard ML and OCaml convention of being bracketed with the lexical tokens (* and *).

Let us now try to import one of the files of the standard library of Matita that you should have downloaded along with the source. Type the following line in the editor window

```
include "basics/logic.ma".
```

and then hit `Ctrl+Alt+Page Down` or press the button at the top of the Matita window with a downarrow on it. If everything is working right, the bottom right-hand pane will start printing out numerous messages, telling you that the system is (recursively) typechecking and including *basics/logic.ma* and its dependencies. When the include command has been completely processed, the command line of the editor pane will turn a light shade of blue. Lines highlighted with this colour are “locked” and cannot be edited any longer: to edit them, you must first ask the system to retract them. You may use `Ctrl+Alt+Page Up` (or the button with an uparrow) to retract a single statement, and `Ctrl+Alt+Home` (or the button with an overlined uparrow) to retract everything in the current file.

If the execution of the command fails, Matita will report its diagnosis in the bottom right hand pane; in the case of the include command, the most frequent reason for a failure is that the system has not been able to find the requested file. If this happens, please check that the “root” file of the previous section has been correctly created and saved in `$SCRIPTS_HOME`.

0.4 Browsing the library

For the moment, it is not very important to know what has been loaded by the system including the file *logic.ma*; however, if you are curious to have a look at its content, the best choice is to directly open it. To this purpose, click `File > Open` in the menu bar: this will open a new pop up window allowing you to browse the file system. Choose the desired file (e.g. *logic.ma*) and confirm your choice by pressing the OK button: the file will open up in the editing window.

Sometimes, you are not interested in reading an entire file, but just to "check" a specific result. For instance, after having included *logic.ma* as described in the previous section, type the following line (without a fullstop) and execute it as usual, by hitting `Ctrl+Alt+Page Down`

```
check True
```

a new window (called Matita Browser) should pop up with type information about the constant True. In particular, this window should tell you that:

```
True : Prop
```

The language of Matita is strongly typed: every term of the language has a type (including types themselves). The syntax *M*:*T* is the standard notation adopted in type theory to express the fact that the term *M* has type *T*. So, the previous statement informs us that *True* is a term of type *Prop*. *Prop* (that is a shorthand for Proposition) is a primitive constant of the system, that stands for the type of all propositions[3]. Hence, the sentence *True* : *Prop* simply affirms that *True* is a proposition.

So, this is the type of *True*, but what is its actual definition? The Matita Browser (similarly to the goal window) is hypertextual: you can directly jump to the definitions of objects by just clicking on them. If you click on *True* you will discover that it is an instance of an entity called *inductive type*. Matita is actually based on a Dependent Type System known as the Calculus of Inductive Constructions (see [23, 18]): in such a framework, as we shall explain in Section 3, almost everything is defined as an inductive type (we shall also come back, in that occasion, to the definition of *True*).

We claimed that every term has a type, so you might wonder what is the type of *Prop*. If you check it, you will discover that

```
Prop : Type[0]
```

In the same way as *Prop* is the type of all propositions, *Type* is, in a sense, the type of all types. But what is the meaning of the label 0? Well, the point is that, to avoid logical paradoxes "à la Russell", types of types (called universes) should be organized into a hierarchy of the following kind

$$\mathit{Type}[0] : \mathit{Type}[1] : \mathit{Type}[2] : \mathit{Type}[3] \ldots$$

For the moment, you may just ignore the existence of *Type*[*i*] for *i* larger than 0.

0.5 Live DVD

If you are not running Linux or you do not want to install Matita, you can download a Live DVD image from the download page of Matita. The image can be burned to a bootable DVD or it can be directly executed in a virtual machine using any virtual machine emulator like VMWare. The image is a full Debian installation

[3]We shall give more details on Matita system of types and its logical restrictions in Section 9

with a copy of Matita already installed. It is sufficient to open a terminal and type `matita` to start experimenting.

0.6 Matita Web

Still another alternative is to interact with Matita on line, through our web interface [3]. Matita is available as a multi-user web application running remotely on our server. The web application can process the same proof scripts as the stand-alone system, adding support for scripts containing HTML-like markup.

Every Matitaweb user has a separate space for storing definitions and proofs. The personal copies can then be synchronized with the centralized library for collaborative developments (selected users, currently testing only).

In order to get access to Matita Web, follow the instructions at the following url: `http://matita.cs.unibo.it/matitaweb.shtml`.

You will also find an on-line, executable version of this tutorial.

1. DATA TYPES, FUNCTIONS AND THEOREMS

Matita is both a programming language and a theorem proving environment: you can define datatypes and programs, and then prove properties about them. Very few things are built-in: not even booleans or logical connectives, but you may of course freely include and reuse libraries, as in normal programming languages. The main philosophy of the system is to let you define your own data-types and functions using a powerful computational mechanism based on the declaration of inductive types.

Let us start this tutorial with a simple example based on the following well known problem.

1.1 The goat, the wolf and the cabbage

A farmer needs to transfer a goat, a wolf and a head of cabbage across a river, but there is only one place available on his boat. Furthermore, the goat will eat the cabbage if they are left alone on the same bank, and similarly the wolf will eat the goat. The problem consists in bringing all three items safely across the river.

Let us start with including the file *logic.ma* that contains a few preliminary notions not worth discussing for the moment. As a general practice, it is advisable to always include *basics/pts.ma* that contains the declaration of universes, or, at least, to include *basics/core_notation.ma* for some basic notations. The former includes the latter, and every file from the standard library recursively includes both.

```
include "basics/logic.ma".
```

Our first data type defines the two banks of the river, which will be named east and west. It is a simple example of enumerated type, defined by explicitly declaring all its elements. The type itself is called “bank”. Let’s have a look at the definition, then we shall discuss its syntax.

```
inductive bank: Type[0] :=
| east : bank
| west : bank.
```

The definition starts with the keyword **inductive** followed by the name we want to give to the new datatype (in this case, *bank*), followed by its type (a type of a type is traditionally called a *sort* in type theory). A sort in Matita is either *Prop* or a *Type*[*i*]. As we already said in the introduction, *Prop* is meant for propositions, *Type*[0] for datatypes and *Type*[*i*] for large collections.

The definition proceeds with the keyword “`:=`” (or `\def`) followed by the *body* of the definition. The body is just a list of *constructors* for the inductive type, separated by the symbol | (vertical bar). Each constructor is a pair composed by a name and its type. Constructors (in our case, *east* and *west*) are the *canonical inhabitants* of the inductive type we are defining (in our case, *bank*), hence their type must be of type *bank*. In general, as we shall see, constructors for an inductive type *T* may have a more complex structure, and in particular can be recursive: the general proviso is that they must always *return* a result of type *T*. Hence, the declaration of a constructor *c* for and inductive type *T* has the following typical shape[4]:

$$c:\ A_1 \rightarrow \ldots \rightarrow A_n \rightarrow T$$

where $A_1 \ldots A_n$ can be arbitrary types, comprising *T* itself.

As a general rule, the inductive type must be conceptually understood as the *smallest* collection of terms *freely generated* by its constructors.

1.2 Defining functions

We can now define a simple function computing, for each bank of the river, the opposite one.

```
definition opposite :=λs.
match s with
  [ east ⇒ west
  | west ⇒ east
  ].
```

Non-recursive functions must be defined in Matita using the keyword **definition** followed by the name of the function and an *optional* type. The type *bank* → *bank* is omitted in the example because it is automatically inferred by the system. The definition proceeds with the keyword "`:=`" followed by the function *body*. The body starts with a list of *input parameters*, preceded by the symbol λ (`\lambda`); many TEX-like macros are automatically converted by Matita into Unicode symbols: see `View > TeX/UTF-8 table` in the menu bar for a complete list.

We then proceed by *pattern matching* on the parameter *s*: if the input bank is *east* we return *west*, and conversely if it is *west* we return *east*. Since the input parameter *s* is matched against the constructors of the type *bank*, its type is inferred to be *bank*. Since in every case the **match** returns a bank, the output of *opposite* is inferred to be *bank* too.

Pattern matching is a well known feature typical of functional programming (especially of the ML family), allowing simple access to the components of complex data structures. A function definition most often corresponds to pattern matching on one or more of its parameters, allowing the function to be easily defined by cases.

The syntactic structure of a match expression is the following:

```
match expr with
  [ p1 ⇒ expr1
  | p2 ⇒ expr2
  :
  | pn ⇒ exprn
  ]
```

The expression *expr*, which is supposed to be an element of an inductive type *T*, is matched sequentially to the various patterns $p_1, \ldots, p_n$. A pattern p_i is just a constructor of the inductive type *T* possibly applied to a list of variables, bound

[4]The notion of inductive type is more general and admits other shapes. They will be discussed in Section 3. Moreover, not every form of recursive constructor is accepted, since, in order to ensure logical consistency, we must respect some *positivity* conditions that we shall discuss in Section 9.

inside the corresponding branch *expr*$_i$. If the pattern *p*$_i$ matches the value *expr*, then the corresponding branch *expr*$_i$ is evaluated (after replacing in it the pattern variables with the corresponding subterms of *expr*). Usually, all expressions *expr* have a single, uniform type; however, since Matita supports *dependent types*, the type of branches could depend on the matched expression, too (see section 5.10).

1.3 Our first lemma

Functions are live entities, and can be actually computed. To check this, let us state the property that the opposite bank of east is west; every lemma needs a name for further reference, so we will call this one `east_to_west`.

```
lemma east_to_west : opposite east = west.
```

If you enter the previous declaration and execute it, you will see a new pane replacing the matita logo on the right side of the screen: it is the goal pane, providing a sequent-like representation of the following form, for each open goal in the proof

$$\frac{\begin{array}{c}B_1\\ B_2\\ \ldots\\ B_k\end{array}}{A}$$

A is the conclusion of the goal and B_1, ..., B_k are the hypotheses in the context. In our case, we only have one goal, and the context is initially empty:

$$\frac{}{\texttt{opposite east = west}}$$

We proceed in the proof by issuing commands (traditionally called tactics) to the system. In this case, we want to evaluate the function, which can be done by invoking the "normalize" command (remember that strings within the delimiters "`(*`" and "`*)`" are just comments):

```
normalize (* this command reduces the goal to the normal form *)
```

By executing it, you will see that the goal will change to `west = west`: in particular, the subexpression `opposite east` has been reduced to `west`. You may use the retract button to undo the step, if you wish.

The new goal `west = west` is trivial: it is just a consequence of reflexivity. Such trivial steps can be closed in Matita by just typing a double slash. We complete the proof by the qed command, that instructs the system to store the lemma performing some book-keeping operations.

```
// qed. (* close the goal by invoking automation
          and add the theorem to the library *)
```

In exactly the same way, we can prove that the opposite side of west is east. In this case, we avoid the unnecessary simplification step: `//` will take care of it.

```
lemma west_to_east : opposite west = east.
// qed.
```

Instead of **lemma**, you may also use **theorem** or **corollary**. Matita does not attempt to make a semantic distinction between them, and their use is entirely up to the user.

In some cases, the **normalize** tactic is too aggressive since the normal form of a term can be very large and unreadable. An alternative is the **whd** tactic that reduce terms only at the top level. Moreover, we will introduce *patterns* in Section 2.9 to better control reduction by restricting it to chosen subterms.

1.4 Introducing hypothesis in the context

A slightly more complex problem consists in proving that opposite is idempotent

```
lemma idempotent_opposite : ∀x. opposite (opposite x) = x.
```

We start the proof by moving *x* from the conclusion into the context, that is a (backward) introduction step. Matita syntax for an introduction step is simply the sharp character "**#**" followed by the name of the item to be moved into the context. This also allows us to rename the item, if needed: for instance if we wish to rename *x* to *b* (since it is a bank), we just type **#***b*.

```
#b  (* introduce b into the context *)
```

After executing this command, the goal-pane will look like the following:

```
b: bank
─────────────────────────
opposite (opposite b) = b
```

The variable *b* has been added to the context, replacing *x* in the conclusion; moreover its implicit type "bank" has been made explicit. The foundational language of Matita is strongly typed, hence every time you declare a variable with some binding mechanism you are supposed to provide its type. Luckily, in many cases, this type can be automatically inferred by the system according to the usage of variable, sparing the user the burden to write it.

1.5 Case analysis

But how are we supposed to proceed, now? Simplification cannot help us, since *b* is a variable: just try to call normalize and you will see that it has no effect. The point is that we must proceed by case-analysis according to the possible values of *b*, namely east and west. To this aim, you must invoke the **cases** command, followed by the name of the hypothesis (more generally, an arbitrary expression) that must be the object of the case analysis (in our case, *b*). Note that we do not need to specify the possible cases: the system is able to compute them from the type of the expression (that must be an inductive type).

```
cases b (* cases analysis on b *)
```

This is the first example of a tactic that takes an argument. In this case the argument is just a variable. In case of a compound expression, parentheses are needed around it.

Executing the previous command has the effect of opening two subgoals, corresponding to the two cases `b=east` and `b=west`: you may view one or the other by clicking on the tabs within the goal pane. Both goals can be closed by trivial computations, so we may use `//` as usual. If we had to treat each subgoal in a different way, we should have *focused* on each of them in turn, in a way that will be described at the end of this section.

```
// qed. (* this command closes both goals *)
```

1.6 Predicates

Instead of working with functions, it is sometimes convenient to work with predicates. For instance, instead of defining a function computing the opposite bank, we could declare a predicate `opp` stating when two banks are opposite to each other; `opp` is a binary predicate on banks, that is, in curried form, an object of type `bank → bank → Prop`. Only two cases are possible, leading naturally to the following inductive definition:

```
inductive opp : bank → bank → Prop :=
| east_west : opp east west
| west_east : opp west east.
```

In precisely the same way as `bank` is the smallest type containing `east` and `west`, `opp` is the smallest predicate containing the two sub-cases `east_west` and `weast_east`. If you have some familiarity with Prolog, you may look at `opp` as the predicate defined by the two clauses - in this case, the two facts - `east_west` and `west_east`.

Between opp and opposite, the following relation holds:

$$opp\ a\ b \leftrightarrow a = opposite\ b$$

Let us prove it, starting from the left to right implication, first.

```
lemma opp_to_opposite: ∀a,b. opp a b → a = opposite b.
```

We start the proof introducing `a`, `b` and the hypothesis `opp a b`, that we call `oppab`. Next, we proceed by case-analysis on the possible proofs of `opp a b` (i.e. on the possible shapes of `oppab`. By definition, there are only two possibilities, namely `east_west` or `west_east`. Both resulting subcases are trivial, and can be closed by automation. The whole proof is hence:

```
#a #b #oppab cases oppab // qed.
```

1.7 Rewriting

Let us consider the opposite direction.

```
lemma opposite_to_opp: ∀a,b. a = opposite b → opp a b.
```

As usual, we start introducing `a`, `b` and the hypothesis `a = opposite b`, that we call `eqa`. The right way to proceed, now, is by *rewriting* `a` into `opposite b`. We do this by typing `>eqa` that instructs the system to rewrite inside the goal using the equation named `eqa` oriented from left to right. If we wished to rewrite in the opposite direction, namely `opposite b` into `a`, we would have typed `<eqa`. In section 2.9 we shall explain how to restrict/localize rewriting (and other operations) by means of *patterns*.

After rewriting, we simply conclude the proof by case-analysis on `b`. Here is the whole proof:

```
#a #b #eqa >eqa cases b // qed.
```

1.8 Records

It is time to proceed with our formalization of the farmer's problem. A state of the system is defined by the position of four items: the goat, the wolf, the head of cabbage, and the boat. The simplest way to declare such a data type is to use a record.

```
record state : Type[0] :={
   goat_pos : bank
 ; wolf_pos : bank
 ; cabbage_pos: bank
 ; boat_pos : bank}.
```

When you define a record named `foo`, the system automatically defines a record constructor named `mk_foo`. To create a new record you need to pass as arguments to `mk_foo` the values of the record fields:

```
definition start :=mk_state east east east east.
definition end :=mk_state west west west west.
```

We must now define the possible moves. A natural way to do it is in the form of a relation (a binary predicate) on states.

```
inductive move : state → state → Prop :=
| move_goat: ∀g,g1,w,c.
    opp g g1 → move (mk_state g w c g) (mk_state g1 w c g1)
    (* We can move the goat from a bank g to the opposite bank g1
       if and only if the boat is on the same bank g as the goat
       and we move the boat along with it. *)
| move_wolf: ∀g,w,w1,c.
    opp w w1 → move (mk_state g w c w) (mk_state g w1 c w1)
| move_cabbage: ∀g,w,c,c1.
    opp c c1 → move (mk_state g w c c) (mk_state g w c1 c1)
| move_boat: ∀g,w,c,b,b1.
    opp b b1 → move (mk_state g w c b) (mk_state g w c b1).
```

We say that a state is *safe* if either the goat is on the same bank of the boat, or both the wolf and the cabbage are on the opposite bank of the goat.

```
inductive safe_state : state → Prop :=
| with_boat : ∀g,w,c.safe_state (mk_state g w c g)
| opposite_side : ∀g,g1,b.opp g g1 → safe_state (mk_state g g1 g1 b).
```

Finally, a state *y* is reachable from *x* if either there is a single move leading from *x* to *y*, or there is a safe state *z* such that *z* is reachable from *x* and there is a move leading from *z* to *y*.

```
inductive reachable : state → state → Prop :=
| one : ∀x,y.move x y → reachable x y
| more : ∀x,z,y. reachable x z → safe_state z → move z y → reachable x y.
```

1.9 Automation

Remarkably, Matita is now able to solve the farmer problem by itself, provided you allow automation to exploit enough resources. The command `/n/` is similar to `//`, where *n* is a measure of this complexity: in particular it is a bound to the depth of the expected proof tree (more precisely, to the number of nested applicative nodes). In this case, there is a solution in six moves, and we need a few more applications to handle reachability, and side conditions. The magic number to let automation work is, in this case, 9.

```
lemma problem: reachable start end.
normalize /9/ qed.
```

The reader is referred to [9, 10] for technical information on Matita's automation facilities.

1.10 Application

Let us now try to derive the proof in a more interactive way. Of course, we expect to need several moves to transfer all items from a bank to the other, so we should start our proof by applying *more*. Matita syntax for invoking the application of a lemma or theorem named *foo* is to write *@foo*. In general, the philosophy of Matita is to describe each proof of a property *P* as a structured collection of objects involved in the proof, prefixed by simple modalities (#,<,@,+...) explaining the way it is actually used (e.g. for introduction, rewriting, in an applicative step, and so on).

```
lemma problem1: reachable start end.
normalize @more
```

After performing the previous application, we have four open subgoals (note by the way that the type of some goals may depend on the values of other goals):

goal	*type*
X	*state*
Y	*reachable* (*mk_state east east east east*) X
W	*safe_state* X
Z	*move* X (*mk_state west west west west*)

namely, we must provide a state `X` such that `(Y)` it is reachable from start, `(W)` it is safe, and `(Z)` there is a move leading from `X` to `end`. All goals are active, as emphasized by the fact that their title in the goal pane are all bold. Any command typed by the user is *normally applied in parallel to all active goals*, but clearly, in the above case, we must proceed in a different way for each of them. The operation of selecting a goal among the active ones is called *focusing* and is described in the next section.

When applying a tactic, check that there is at least one active goal; otherwise the command will be silently discharged.

1.11 Focusing

The general idea is that when you have multiple subgoals you should structure your proof accordingly, using a syntax like `[`p_1`|`p_2`|`...`|`p_n`]` where you have a subproofs p_i for every active subgoal. The inner proofs can branch again when multiple goals become active, resulting in a tree-like proof structure.

In all other provers, `[`p_1`|`p_2`|`...`|`p_n`]` is a new tactic built from the *tactical* `[`...`|`...`|`...`]`. A tactical builds a tactic from arguments that are tactics (or proofs) themselves. Tactics built with tacticals are executed as a monolitic step: it is not possible to stop in the middle of execution to observe the intermediate proof states.

Matita decomposes the `[`...`|`...`|`...`]` tactical into three commands "`[`", "`|`" and "`]`", called *tinycals* [20], that can be executed individually. More precisely,

—the tinycal "`[`" opens a new focusing section for the currently active goals, and focus on the first of them

—the tinycal "`|`" shifts the focus to the next goal in the current section

—the tinycal "`]`" closes the focusing section, falling back to the previous level and adding to it all the remaining (not yet closed) goals.

Matita also provides other tinycals and a few tacticals that are not described in the tutorial, but can be found in the user manual.

Let us see the effect of the "`[`" on our proof. We were in a state where we had four active goals. By executing "`[`" the four goals get a progressive number, and the first of them (the new intermediate state) gets the focus.

We can now proceed in several possible ways. The most straightforward way is to explicitly supply the next intermediate state, that is `(mk_state east west west east)`. We can do it, by *applying* the term `(mk_state east west west east)`.

This application closes the current goal; at present, no goal has the focus on. In order to act on the next goal, we must focus on it using the "`|`" operator. In this case, we would like to skip the next goal, and focus on the trivial third subgoal: a simple way to do it, is by typing "`|`" again. The current goal is hence:

```
safe_state (mk_state east west west east)
```

whose proof is trivial and can be delegated to automation.

We then advance to the next goal, namely the fact that there is a move from `(mk_state east west west east)}` to `(mk_state west west west west)}`; this is trivial too, but it requires `/2/` (automation at depth two) since `move_goat}` opens an additional subgoal. By applying "]" we refocus on the skipped goal, going back to a situation similar to the one we started with.

Here is this fragment of the proof:

```
[@(mk_state east west west east) || // | /2/ ]
```

Note that we had four goals, we closed three of them, hence we are left with a single goal:

```
─────────────────────────────────────────────────────────────
reachable (mk_state east east east east) (mk_state east west west east)
```

1.12 Implicit arguments and partial instantiation

Let us perform the next step, namely moving back the boat, in a slightly different way. The `more` operation expects as its second argument the new intermediate state, hence instead of applying `more` we can apply this term *already instantiated* on the next intermediate state, that is

`more ? (mk_state east west west west)`

The question mark stands for an *implicit argument* to be inferred by the system. The joint use of partial instantiation and implicit arguments is a very powerful tool for reducing the length of proofs; Matita's inference system is based on a sophisticated bidirectional algorithm [7], exploiting both exptected and inferred types, that is particularly convenient for the automatic management of implicit information.

By applying the previous term, we get three *independent* subgoals (i.e. not sharing any variable), all active, and two of them are trivial. We can just apply automation to all of them in parallel, and it will close the two trivial goals. In this case, we performed a move of the boat with the following code (note in particular the we had no need to focus):

```
@(more ? (mk_state east west west west)) /2/
```

Let us come to the next step, that consists of moving the wolf. Suppose that instead of specifying the next *intermediate state*, we prefer to specify the next *move*. In the spirit of the previous example, we can do it in by simply instantiating `more` with the suitable move, that is `(more ... (move_wolf ...))`. The dots stand here for an arbitrary number of implicit arguments, to be guessed by the system, and can be typed in Matita as `\ldots` .

Unfortunately, the previous move is not enough to fully instantiate the new intermediate state, and we obtain the following goals:

goal	*type*
X	`reachable (mk_state east east east east) (mk_state east W west W)`
Y	`safe_state (mk_state east W west W)`
Z	`opp W west`
W	`bank`

In particular, the bank towards which we move (`W`) remains unknown (we know that it is the opposite of the current one (`Z`), but this information is still implicit).

Automation cannot help here, since all goals depend from the bank `W` (they are in a same *cluster*, in Matita terminology) and automation refuses to close some subgoals instantiating other subgoals remaining open (the instantiation could be arbitrary). The simplest way to proceed is to focus on the bank, that is the *fourth* subgoal, and explicitly instantiate it. Instead of repeatedly using "`|`", we can perform focusing by typing "`4:`" as described by the following command.

```
[4: @east] /2/
```

Alternatively, we can directly instantiate the bank into the move. Let us complete the proof in this, very readable way.

```
@(more ...(move_goat west ...)) /2/
@(more ...(move_cabbage ?? east ...)) /2/
@(more ...(move_boat ??? west ...)) /2/
@one /2/ qed.
```

2. INDUCTION

Most of the types we have seen so far have been enumerated types, composed of a finite set of alternatives, and records, composed of tuples of heterogeneous elements. A more interesting case of type definition is when some of the rules defining its elements are recursive, i.e. they allow the formation of more elements of the type in terms of the already defined ones.

The most typical case is provided by the natural numbers, which can be defined as the smallest set generated by a constant 0 and a successor function from natural numbers to natural numbers:

```
inductive nat : Type[0] :=
| O : nat
| S : nat → nat.
```

The two terms `O` and `S` are called constructors: they define the signature of the type, whose objects are the elements freely generated by means of them. So, examples of natural numbers are `O, S O, S (S O), S (S (S O))` and so on.

The language of Matita allows the definition of well founded recursive functions on inductive types; in order to guarantee termination of recursion you are only allowed to make recursive calls on arguments structurally smaller than the ones you received as input. Most mathematical functions can be naturally defined in this way. For instance, the sum of two natural numbers can be defined as follows

```
let rec add n m :=
 match n with
 [ O ⇒ m
 | S a ⇒ S (add a m)
 ].
```

Observe that the definition of a recursive function requires the keyword `let rec` instead of `definition`. The specification of formal parameters is different too. In this case, they come before the body, and do not require a λ. If you need to specify the type of some argument, you need to enclose it in parentheses, e.g. `(n:nat)`.

By convention, recursion is supposed to operate on the first argument that means that this is the only argument that is supposed to decrease in the recursive calls. In case you need to work on a different argument, say foo, you can specify it by explicitly mentioning "`on foo`" just after the declaration of all parameters.

2.1 Elimination

As we remarked at the end of the previous section, the function "`add`" works by recursion on the first argument. This means that, for instance, `(add O x)` will reduce to `x`, as expected, but the computation of `(add x O)` is stuck. How can we prove that, for a generic `x`, `add x O = x`? The mathematical tool to do this is called *induction*. The induction principle for natural numbers states that, given a property $P(n)$, if we prove $P(O)$ and prove that, for any m, $P(m)$ implies $P(S\,m)$, then we can conclude $P(n)$ for any n.

The "`elim`" tactic allows us to apply induction in a very simple way. If the goal is of the form `P n`, the invocation of

```
elim n
```

will break it down to the two subgoals `P 0` and `∀m.P m → P (S m)`. Let us apply it to our case

```
lemma add_O: ∀a. add a O = a.
#a elim a
```

We generated the following goals:

goal	*type*
G_1	`add O O = O`
G_2	`∀x. add x O = x → add (S x) O = S x`

After normalization, both goals are trivial:

```
normalize // qed.
```

In a similar way, it is convenient to state a lemma about the behaviour of add when the second argument is not zero; the proof is a simple induction on `a`:

```
lemma add_S : ∀a,b. add a (S b) = S (add a b).
#a #b elim a normalize // qed.
```

We are now in the position to prove the commutativity of `add`. We proceed by induction on the first argument, and simplify the goals by invoking normalization:

```
theorem add_comm : ∀a,b. add a b = add b a.
#a elim a normalize
```

We are left with two sub goals:

goal	*type*
G_1	`∀b. b = add b O`
G_2	`∀x.(∀b. add x b = add b x) →∀b. S (add x b) = add b (S x)`

G_1 is just our lemma `add_O`. For G_2, we could start introducing `x` and the induction hypothesis `IH`; then, the goal would be proved by rewriting using `add_S` and `IH`. The resulting script would be `#x #IH >add_S >IH //`. Instead, this easy proof can be found automatically by Matita invoking the automation tactic `//`.

2.2 Existentials

We are interested in proving that for any natural number `n` there exists a natural number `m` that is the *integer half* of `n`, defined as the result of the integer division of `n` by 2. This will give us the opportunity to introduce new connectives and quantifiers and, later on, to highlight some important aspects of proofs and computations. It is interesting to observe that not even existential quantification is a primitive notion in Matita: in fact, it is defined in the library file `basic/logic.ma`, and in order to use it we need to include this file, or another one that recursively includes it (we shall come back on the actual definition of the existential in section 3.2). Here is the formal statement of the theorem we are interested in:

```
include "basics/types.ma".

definition twice :=λn.add n n.

theorem ex_half: ∀n.∃m. n = twice m ∨ n = S (twice m).
```

We proceed by induction on `n`; after normalizing, we get the following goals:

goal	*type*
G_1	`∃m.O = add O O ∨ O = S (add m m)`
G_2	`∀x.(∃m. x = add m m ∨ x = S (add m m)) →`
	`∃m. S x = add m m ∨ S x = S (add m m)`

The only way we have to prove an existential goal is by exhibiting the witness, which in the case of the first goal is O. We do it by applying the term called *ex_intro* instantiated by the witness. Then, it is clear that we must follow the left branch of the disjunction. In Section 3.2 we will explain that the disjuction connective is defined as an inductive type generated by two constructors, called *or_introl* and *or_intror*. Therefore to proceed in the proof we can apply the term *or_introl*. However, remembering the names of constructors can be annoying: we can invoke the application of the n-th constructor of an inductive type (inferred from the current goal) by typing $\%n$. At this point, we are left with the subgoal $O = add\ O\ O$, that is closed by computation. It is worth to observe that invoking automation at depth /3/ would also automatically close G_1. Here is the fragment of the proof, up to this point:

```
#n elim n normalize
  [@(ex_intro ...O) %1 //
```

2.3 Decomposition

The case of G_2 is more complex. We should start introducing `x` and the induction hypothesis

`IH: ∃m. x = add m m ∨ x = S (add m m)`

The induction hypothesis `IH` asserts the existence of an `m` that satisfies the thesis; to obtain such an `m`, we eliminate the existential at the front of the hypothesis using the case tactic. This is motivated by the fact that the existential is just a particular inductive type. This situation, where we introduce something into the context and immediately eliminate it by case analysis is so frequent that Matita provides a convenient shorthand: you can just type a single "`*`". The star symbol should be reminiscent of an explosion: the idea is that you have a structured hypothesis, and you ask to explode it into its constituents. In the case of the existential, it allows us to pass from a goal of the shape

`(∃x.P x) → Q`

to a goal of the shape

`∀x.(P x → Q)`

At this point we are left with a new goal with the following shape

$$G_3: \forall m.\ (x = add\ m\ m \lor x = S\ (add\ m\ m)) \to \ldots$$

We should introduce m, the hypothesis H: $x = add\ m\ m \lor x = S\ (add\ m\ m)$, and then reason by cases on this hypothesis. It is the same situation as before: we must explode the disjunctive hypothesis into its constituents. In the case of a disjunction, the * tactic allows us to pass from a goal of the form

$$A \lor B \to Q$$

to two subgoals of the form

$$A \to Q \text{ and } B \to Q$$

In the first subgoal, we are under the assumption that $x = add\ m\ m$. Half of $(S\ x)$ is therefore m, and we have to prove the right branch of the disjunction. In the second subgoal, we are under the assumption that $x = S\ (add\ m\ m)$. The half of $(S\ x)$ is hence $(S\ m)$, and we have to follow the left branch of the disjunction.
Here is the simple proof of G_2:

```
  |#x * #m * #eqx
    [@(ex_intro ...m) /2/ | @(ex_intro ...(S m)) normalize /2/
  ]
qed.
```

2.4 Computing vs. Proving

Instead of proving the existence of a number corresponding to the half of n, we could be interested in computing it. The best way to do it is to define this division operation together with remainder, that in our case is just a boolean value: true if the input term is even, and false if the input term is odd. The boolean data type is defined in *basics/bool.ma*: it is just an inductive type with two constructors *true* and *false*.

Since we must return a pair, we could use a suitably defined record type, or simply a product type $nat \times bool$, already defined in the basic library in *basics/types.ma*, together with notation for building and destructuring pairs. The product type is just a sort of "general purpose" record, with standard fields fst and snd, called projections. A pair of values n and m is written $\langle n,m \rangle$. When p is a pair, the expression **let** $\langle x,y \rangle$:=p **in** E binds x and y in E respectively to the first and second component of p.

We first write down the division function, and then discuss it.

```
let rec div2 n :=
match n with
[ O ⇒ ⟨O,false⟩
| S a ⇒
   let ⟨q,r⟩ :=(div2 a) in
   match r with
   [ true ⇒ ⟨S q,false⟩
   | false ⇒ ⟨q,true⟩
   ]
].
```

It is important to point out the substantial analogy between the algorithm for computing *div2* and the proof of *ex_half*. Consider *ex_half* which returns a proof of the form $\exists n.A\ n \lor B\ n$: the real informative content in it is the witness *n* and a boolean indicating which one between the two conditions *A n* and *B n* is met. This is precisely the quotient-remainder pair returned by *div2*. In both cases we recur (respectively, by induction or by recursion) over the input argument *n*. In case *n* = 0, we conclude the proof in *ex_half* by providing the witness *O* and a proof of *A O*; this corresponds to returning the pair ⟨*O*,*false*⟩ in *div2*. Similarly, in the inductive case *n* = *S a*, we must exploit the inductive hypothesis for a (i.e. the result of the recursive call), distinguishing two subcases according to the two possibilities *A a* or *B a* (i.e. the two possible values of the remainder for *a*). The reader is invited to check all the details of this correspondence.

2.5 Destruct

Let us now prove that our *div2* function has the expected behaviour. We start proving a few easy lemmas:

```
lemma div2SO: ∀n,q. div2 n = ⟨q,false⟩ → div2 (S n) = ⟨q,true⟩.
#n #q #H normalize >H normalize // qed.

lemma div2S1: ∀n,q. div2 n = ⟨q,true⟩ → div2 (S n) = ⟨S q,false⟩.
#n #q #H normalize >H normalize // qed.
```

Here is our statement, where *nat_of_bool* is the conversion function that maps false to zero and true to one:

```
lemma div2_ok: ∀n,q,r. div2 n = ⟨q,r⟩ → n = add (nat_of_bool r) (twice q).
```

We proceed by induction on *n*, which produces two subgoals. The first subgoal looks like the following:

```
∀q,r.div2 O = ⟨q,r⟩ → O = add (nat_of_bool r) (twice q)
```

We may introduce *q*,*r* and the hypothesis

```
H: div2 O = ⟨q,r⟩
```

Note that the left hand side of this equation is not in normal form, and we would like to reduce it. We can do it by specifying a pattern for the normalize tactic, introduced by the "**in**" keyword, and delimited by a semicolon. In this case, the pattern is just the name of the hypothesis, and we should write

```
normalize in H
```

At this point, the first subgoal looks like the following:

```
n: nat
q: nat
r: bool
H: ⟨O, false⟩=⟨q, r⟩
________________________________
O = add (nat_of_bool r) (twice q)
```

From the hypothesis *H* we expect to be able to conclude that *q=O* and *r=false*. The tactic that provides this functionality is called `destruct`. The tactic decomposes every hypothesis that is an equality between structured terms into smaller components: if an absurd situation is recognized (like an equality between *O* and (*S n*), built from different constructors) the current goal is automatically closed; otherwise, all derived equations where one of the sides is a variable are automatically substituted in the proof, and the remaining equations are added to the context (replacing the original equation). The tactic considers the two sides to be structured if they are the application of constructors, and skips equalities involving at least one function application (e.g. *f n = S m*), unless the other side is a variable (e.g. *f n = m*, handled by substituting *f n* for *m* everywhere).

In the above case, by invoking destruct we would get the two equations *q=O* and *r=false*; these are immediately substituted in the goal, that becomes:

```
n: nat
q: nat
r: bool
________________________________________
O = add (nat_of_bool false) (twice O)
```

and can be solved by computation.

2.6 Cut

The next subgoal, after performing a few introductions, looks like the following:

```
n: nat
a: nat
Hind: ∀q:nat.∀r:bool.div2 a=⟨q,r⟩→ a=add (nat_of_bool r) (twice q)
q: nat
r: bool
______________________________________________________________
div2 (S a) = ⟨q,r⟩→ S a = add (nat_of_bool r) (twice q)
```

We should proceed by case-analysis on the remainder of (*div2 a*), but before doing it we should emphasize the fact that (*div2 a*)} can be expressed as a pair of its two projections. The tactic that allows to introduce a new hypothesis, splitting complex proofs into smaller components is called *cut*. The invocation of `cut` *A* transforms the current goal *G* into the two subgoals *A* and *A* → *G* (*A* is called the cut formula).

In our case, the cut formula is

```
div2 a = ⟨fst ...(div2 a), snd ...(div2 a)⟩
```

whose proof is trivial. After typing the following commands

```
cut (div2 a = ⟨fst ...(div2 a), snd ...(div2 a)⟩) [//]
```

we are left in the new state:

```
n: nat
a: nat
Hind: ∀q:nat.∀r:bool.div2 a=⟨q,r⟩→ a=add (nat_of_bool r) (twice q))
q:nat
r:bool
────────────────────────────────────────
div2 a=⟨fst ...(div2 a),snd ...(div2 a)⟩→
      div2 (S a)=⟨q,r⟩→ S a=add (nat_of_bool r) (twice q)
```

We now proceed by induction on (*snd* ...(*div2 a*)); the two subgoals are respectively closed using the two lemmas *div2SO* and *div2S1* in conjunction with the inductive hypothesis, and do not contain additional challenges.

The *whole* proof of *div2_ok* is therefore the following:

```
#n elim n
  [#q #r #H normalize in H; destruct //
  |#a #Hind #q #r
   cut (div2 a = ⟨fst ...(div2 a), snd ...(div2 a)⟩) [//]
   cases (snd ...(div2 a)) #H
    [>(div2S1 ...H) #H1 destruct normalize @eq_f >add_S @(Hind ...H)
    |>(div2SO ...H) #H1 destruct normalize @eq_f @(Hind ...H)
    ]
qed.
```

2.7 Lapply

The cut rule is the main tool for "forward reasoning" in interactive provers, that is for introducing a new intermediate proposition *P* between the given hypothesis and the current goal.

If we already have in mind an explicit proof *H* for the proposition *P*, a viable alternative to introduce *P* with a cut and prove it with *H* is to call the following tactic:

```
lapply H
```

If the type of *H* is *P*, the effect of the tactic is to transform the current goal *G* into the new goal *P* → *G*.

For instance, if *n*:*nat*, the tactic

```
lapply (add_S n O) S
```

would transform a goal *G* into *add n (S O) = S (add n O)* → *G*.

The name **lapply** stands for "left application", where left should be understood in sequent-like sense, that is as referring to the left side of the sequent, which is the context. It is in fact a generalization of the left introduction rule for application in sequent like calculi.

The advantage of **lapply** with respect to a cut is to avoid the explicit writing of the cut-formula, that in the case of **lapply** is automatically inferred, being the type of its argument.

The **lapply** tactic can also be used to generalize hypotheses, e.g. before performing an elimination. If we have a goal *G* in a context where *x*:*A*, the execution of the command

```
lapply x
```

will transform the goal into ∀*x*:*A*.*G*, generalizing the variable *x* in *G*. If *x* is also used in some hypothesis *H*:*P*, the goal is usually generalized on *H* using **lapply** *H* before doing the same on *x*, resulting in ∀*x*:*A*.*P* → *G*. After generalizing on *x*, the command -*x* is often used to erase the old — and now useless — assumption *x*:*A* from the context.

2.8 Mixing proofs and computations

So far we have seen how to prove the existence of the integer half of a natural number, how to compute it by an explicit program and how to specify and prove the correctnes of such a program.

There is still another possibility: to mix the program and its specification into a single entity. The idea is to refine the output type of the *div2* function: it should not just be a generic pair ⟨*q*,*r*⟩ of natural numbers but a specific pair satisfying the specification of the function. In other words, we need the possibility to define, for a type *A* and a property *P* on *A*, the subset type {*a*:*A*|*P a*} of all elements *a* of type *A* that satisfy the property *P*. *Subset types* are just a particular case of *dependent types*, that are types that can depend on arguments, such as arrays of a specified length taken as a parameter. This kind of types is quite unusual in traditional programming languages, and their study is one of the new frontiers of the current research on type systems.

There is nothing special in a subset type {*a*:*A*|*P a*}: it is just a record composed of an element *a* of type *A* and a *proof* of *P a*. Nevertheless, it provides a language rich enough to comprise proofs among its expressions.

```
record Sub (A:Type[0]) (P:A →Prop) : Type[0] :=
  {witness: A; proof: P witness}.

definition qr_spec :=λn.λp.∀q,r. p=⟨q,r⟩ →n=add (nat_of_bool r) (twice q).
```

We can now construct a function from *n* to {*p*|*qr_spec n p*} by composing the objects we already have:

```
definition div2P: ∀n. Sub (nat×bool) (qr_spec n) :=λn.
 mk_Sub ?? (div2 n) (div2_ok n).
```

But we can also try to directly build such an object:

```
definition div2Pagain : ∀n.Sub (nat×bool) (qr_spec n).
#n elim n
  [@(mk_Sub ...⟨O,false⟩) normalize #q #r #H destruct //
  |#a * #p #qrspec
   cut (p = ⟨fst ...p, snd ...p⟩) [//]
   cases (snd ...p) #H
    [@(mk_Sub ...⟨S (fst ...p),false⟩) #q #r #H1
    destruct @eq_f >add_S @(qrspec ...H)
   |@(mk_Sub ...⟨fst ...p,true⟩) #q #r #H1 destruct @eq_f @(qrspec ...H)
  ]
qed.
```

The reader is invited to run the computation of the previous function on a few examples, to see what the output looks like. The only "readable" information is, in fact, the witness: the proof is, in general, a complex lambda term whose only purpose is to provide a mechanically verifiable certificate that the witness satisfies a given property. We shall come back to subset types is section 4.2.

2.9 Tactic patterns

Generally speaking, the scope of a tactic is the conclusion of the current goal (or the current goals if many are selected at once). In section 2.5, however, we have used **normalize in** *H*; to indicate that the tactic should take effect within hypothesis *H* rather than the conclusion. This "**in** *H*;" is an instance of a more general scheme to express *paths* within the current goals by means of *patterns*. Patterns allows us to specify any subterm of the conclusion or of any hypothesis.

Since the purpose of a pattern is to identify a subterm within a term, its syntax is similar to that of a term, where, in addition to the usual syntax, the symbol `%` is used to specify the target position (i.e. the subterm where the tactic should act); all the parts of the term that must not be targeted by the tactic are replaced by `?` symbols in the pattern. For the purpose of matching against patterns, all user defined notation (e.g. infix symbols, omitted terms to be inferred by the system) is ignored. Notations will be discussed in Section 3.3.

For example, if we are given a term *x* = *a* + *b* + *c*, that, ignoring notation for addition and equality, is

```
eq nat x (plus (plus a b) c)
```

we can specify the subterm *a*+*b* by means of the pattern `(???(?%?))`. The pattern specifies that the term is an application of one term (called head of the application) to three arguments: the head and the first two arguments are generic (this is expressed by the `?` symbol), while the last argument is itself an application with one head and two arguments, the former of which should be the scope of a tactic (as expressed by the `%` symbol). Notice that the pattern sublanguage does not understand at all user defined notation, comprising that defined in the standard library: therefore, a pattern in the form `?=%+?` is not legal.

Patterns can be used to identify a path in either a hypothesis or in the conclusion, by means of the following syntaxes:

```
normalize in ⊢(???(?%?)); (* rewrites in the conclusion *)
normalize in H1:(???(?%?)); (* rewrites in hypothesis H1 *)
```

We use `\vdash` to insert the ⊢ symbol (which is required when specifying a subterm in the conclusion).

Patterns are not restricted to applicative terms, but can be used under all circumstances:

—in arrow types: in a term `(x=1 → x>0)`, `(??%? → ?)` specifies the first *x*;

—within binders: in `(∀x,y:nat.P (x*y))`, `(∀_,_:?.?%)` specifies $x * y$ (notice that in the pattern, bound variables can be replaced by the `_` "don't care" placeholder);

—within a **`match`**: **`match`** `? ` **`with`** `[ _ ⇒ % | _ ⇒ ?]` specifies the subterm contained in the first branch of a binary **`match`**. Notice that the names of costructors *must* be replaced by underscores in the pattern.

A second kind of pattern is used to specify all the subterms that match a user-specified term `u`: these are expressed by the syntax "**`in match`** `u;`".

For instance it is possible to match all additions in the conclusion by means of the pattern **`in match`** `(? + ?);`.

Example 1. *The two kinds of patterns can be combined in a single statement. Suppose the current goal has a hypothesis* `H` *of type:*

`∀z:nat.z*(x+y)=z*x+z*y`

Then to match only the leftmost sum, we can use the pattern

```
in match (?+?) in H1:(∀_:?.??%?);
```

This instructs Matita to match additions, but only within the left-hand side of the equality.

3. EVERYTHING IS AN INDUCTIVE TYPE

As we have mentioned several times, very few notions are really primitive in Matita: one of them is the notion of *universal quantification* (or *dependent product type*) and the other one is the notion of *inductive type*. Even the arrow type (also called function space) $A \to B$ is not really primitive: it can be seen as a particular case of the dependent product $\forall x{:}A.B$ in the degenerate case when B does not depends on x. All the other familiar logical connectives - conjunction, disjunction, negation, existential quantification, even equality - can be defined as particular inductive types.

We shall look at those definitions in this section, since it can be useful to acquire confidence with inductive types, and to get a better theoretical grasp on them.

3.1 Conjunction

In natural deduction, logical rules for connectives are divided in two groups: there are *introduction* rules, allowing us to introduce a logical connective in the conclusion, and there are *elimination* rules, describing how to deconstruct information about a compound proposition into information about its constituents (i.e. how to use a hypothesis having a given top-level connective).

Consider conjunction. In order to understand the introduction rule for conjunction, you should answer the question: how can we prove $A \wedge B$? The answer is simple: we must prove both A and B. Hence the introduction rule for conjunction is $A \to B \to A \wedge B$.

The general idea for defining a logical connective as an inductive type is simply to define it as the *smallest proposition* generated by its introduction rule(s).

For instance, in the case of conjunction, we define

```
inductive And (A,B:Prop) : Prop :=
    conj : A → B → And A B.
```

The corresponding elimination rule is induced by minimality: if we have a proof of $A \wedge B$ it may only derive from the conjunction of a proof of A and a proof of B. A possible way to formally express the elimination rule is the following:

$$\forall A,B,P{:}Prop.\ (A \to B \to P) \to A \wedge B \to P$$

that is, for all A and B, and for any proposition P if we need to prove P under the assumption $A \wedge B$, we can reduce it to proving P under the pair of assumptions A and B.

It is interesting to observe that the elimination rule can be easily derived from the introduction rule *in a completely syntactic way*. Basically, the general structure of the (non dependent) elimination rule for an inductive type T is the following

$$\forall \vec{A},P.\ C_1 \to \ldots \to C_n \to T \to P$$

where $\vec{A}$ is the list of parameters of the inductive type, and every C_i is derived from the type T_i of a constructor c_i of T by just replacing T with P in it. For instance, in the case of the conjunction we only have one constructor of type $A \to B \to A \wedge B$ and replacing $A \wedge B$ with P we get $C_1 = A \to B \to P$.

Every time we declare a new inductive proposition or type *T*, Matita automatically declares an axiom called *T_ind*[5], which embodies the elimination principle for this type. The actual shape of the elimination axiom is more complex than the one described above, since it also takes into account the possibility that the predicate *P* depends on the term of the given inductive type (and possible arguments of the inductive type).

Actually, the elimination tactic **elim** for an element of type *T* is a straightforward wrapper that applies the suitable elimination axiom.

3.2 Disjunction, False, True, Existential Quantification

Let us introduce now other connectives as inductive types, mimicking what we did for conjunction and starting with disjunction. The first point is to derive the introduction rule(s). When can we conclude $A \lor B$? Clearly, we must either have a proof of *A*, or a proof of *B*. So, we have two introduction rules, in this case:

$$A \to A \lor B \qquad \text{and} \qquad B \to A \lor B$$

that leads us to the following inductive definition of disjunction:

```
inductive Or (A,B:Prop) : Prop :=
     or_introl : A → Or A B
   | or_intror : B → Or A B.
```

The elimination principle, automatically generated by the system is

$$or\_ind\colon \forall A,B,P{:}Prop.\ (A \to P) \to (B \to P) \to A \lor B \to P$$

that is a traditional formulation of the elimination rule for the logical disjunction in natural deduction: if *P* follows from both *A* and *B*, then it also follows from their disjunction.

More surprisingly, we can apply the same methodology to define the constant *False*. The point is that, obviously, there is no (canonical) way to conclude *False*: so we have no introduction rule, and we must define an inductive type *without constructors*, which is accepted by the system.

```
inductive False: Prop :=.
```

The elimination principle is in this case

$$False\_ind\colon \forall P{:}Prop.False \to P$$

that is the well known principle "ex falso quodlibet".

What about True? You may always conclude True, hence corresponding inductive definition just has one trivial constructor, traditionally named *I*:

```
inductive True: Prop := I : True.
```

As expected, the elimination rule is not informative in this case: the only way to conclude *P* from *True* is to prove *P*:

[5]In case of inductive data types of sort $Type[j]$ for some j, the system also generates for each i an elimination rules towards $P{:}Type[i]$, called *T_rect_Typei*. The rule allows us to inhabit some type $U{:}Type[i]$ by recursion on the eliminated term of type $Type[j]$.

$$True\_ind\colon\ \forall P{:}Prop.P \to True \to P$$

Finally, let us consider the case of existential quantification. In order to conclude $\exists x{:}A.Q\ x$ we need to prove $Q\ a$ for some term $a{:}A$. Hence, the introduction rule for the existential looks like the following:

$$\forall a{:}A.\ Q\ a \to \exists x.Q\ x$$

from which we get the following inductive definition, parametric in A and Q.

```
inductive ex (A:Type[0]) (Q:A → Prop) : Prop :=
    ex_intro: ∀x:A. Q x → ex A Q.
```

In the next Section we will see how to associate the usual notation $\exists x{:}A.Q$ to $ex\ A\ \lambda x.Q$.

The elimination principle automatically generated by Matita is

$$ex\_ind\colon\ \forall A.\forall Q\backslash{:}A \to Prop.\ \forall P{:}Prop.\ (\forall x{:}A.\ Q\ x \to P) \to (\exists x{:}A.Q) \to P$$

That is, if we know that P is a consequence of $Q\ x$ for any $x{:}A$, then it is enough to know $\exists x{:}A.Q\ x$ to conclude P. It is also worth spelling out the backward reading of the previous principle. Suppose we need to prove P under the assumption $\exists x{:}A.Q$. Then, eliminating the latter amounts to assuming the existence of $x{:}A$ such that $Q\ x$ and proceed to prove P under these new hypotheses.

3.3 A bit of notation

Since we make frequent use of logical connectives and quantifiers, it would be nice to have the possibility to use a more familiar-looking notation for them. Matita offers you the possibility to associate your own notation to any notion you define.

To exploit the natural overloading of notation typical of scientific literature, its management in Matita is split in two steps: one from *presentation* level to *content* level, where we associate a *notation* to a fragment of abstract syntax, and one from *content* level to *term* level, where we provide (possibly multiple) *interpretations* for the abstract syntax.

The mapping between the presentation level (i.e. what the user writes as input and what is displayed in the goal panes) and the content level is defined with the **notation** command.

```
notation "hvbox(a break \land b)"
  left associative with precedence 35 for @{ 'and $a $b }.
```

This declaration associates the infix notation `a \land b` (rendered as $a\wedge b$ by a built-in Unicode mapping) with an abstract syntax tree composed by the new symbol `'and` applied to the result of the parsing of input argument `a` and `b`.

The presentation pattern is always enclosed in double quotes. The special keyword `break` indicates the line breaking point and the box schema `hvbox` indicates a horizontal or vertical layout, according to the available space for the rendering (they can be safely ignored for this tutorial).

The content pattern begins right after the for keyword and extends to the end of the declaration. Parts of the pattern surrounded by `@{...}` denote verbatim content fragments, those surrounded by `${...}` denote meta-operators and references to notational meta-variables occurring in the presentation pattern (for example `$a`).

The declaration also informs the system that the notation is supposed to be left associative and provides information about the syntactic precedence of the operator, which governs the way an expression with different operators is interpreted by the system. For instance, suppose we declare logical disjunction at a lower precedence:

```
notation "hvbox(a break \lor b)"
  left associative with precedence 30 for @{ 'or $a $b }.
```

Then, an expression like `A ∧ B ∨ C` will be understood as `(A ∧ B)∨ C` and not as `A ∧ (B ∨ C)`.

An annoying aspect of user defined notation is that it will eventually interfere with the primitive one, so introducing operators with suitable precedence is an important and delicate issue. The best thing to do is to consult the file basics/core_notation.ma and, unless you cannot reuse an already existing notation overloading it (which is the recommended solution), try to figure out the most suitable precedence for your notation by analogy with other notations. Another possibility is to use the `Terms grammar` entry of the `View` menu of Matita to look at all notations currently loaded, partitioned according to their precedence level.

The next step is to associate an interpretation with content patterns, in the following way:

```
interpretation "logical and" 'and x y = (And x y).

interpretation "logical or" 'or x y = (Or x y).
```

With these commands we are saying that a possible interpretation of the symbol `'and` is the inductive type `And`, and a possible interpretation of the symbol `'or` is the inductive type `Or`. But we could e.g. define boolean functions `andb` and `orb`, and provide an alternative interpretation of `'and` and `'or` on them. As a result, the notations `A∧ B` and `A∨ B` would be overloaded, and interpreted as a proposition or a boolean depending on the context they occur in. For details on the resolution of overloading, see [?].

Let us conclude this section with a discussion of the notation for the existential quantifier. In contrast to what we did previously, we use one mapping for the notation used to pretty print terms, and a different one for parsing. The one for parsing will be more complex to allow omission of types and packing of multiple declarations into a more compact syntax.

```
notation < "hvbox(\exists ident i : ty break . p)"
  right associative with precedence 20 for @{'exists (λ${ident i}: $ty. $p)}.
```

The `<` symbol after the `notation` keyword states that this mapping will only be used during pretty printing. The other main novelty is the special keyword `ident` that instructs the system that the variable `i` is expected to be an identifier. Matita abstract syntax trees include lambda terms as primitive data types, and the previous declaration simply maps the notation `∃x:T.P` into a content term of the form `('exists (λx:T'.P'))` where `T'` and `P'` are the content term obtained from `T` and `P`.

The interpretation is then straightforward:

```
interpretation "exists" 'exists x = (ex ? x).
```

Our notational language has an additional list operator for dealing with variable-sized terms having a regular structure. This operator has a corresponding fold operator, used to build up trees at the content level.

For example, in the case of quantifiers, it is customary to group multiple variable declarations under a same quantifier, writing e.g. $\exists x,y,z.P$ instead of $\exists x.\exists y.\exists z.P$.

This can be achieved by the following notation, used only during parsing as specified by the > symbol:

```
notation > "\exists list1 ident x sep , opt (: T). term 19 Px"
 with precedence 20
 for ${ default
        @{ ${ fold right @{$Px} rec acc @{'exists (λ${ident x}:$T.$acc)} } }
        @{ ${ fold right @{$Px} rec acc @{'exists (λ${ident x}.$acc)} } }
     }.
```

The presentational pattern matches terms starting with the existential symbol, followed by a list of identifiers separated by commas, optionally terminated by a type declaration, followed by a fullstop and finally by the body term. We use *list1* instead of *list0* since we expect to have at least one identifier: conversely, you should use *list0* when the list can possibly be empty.

The *default* meta operator at content level, matches the presentational *opt* and has two branches, which are chosen depending on the matching of the optional subexpression. Let us consider the first, case, where we have an explicit type. The content term is built by folding the function

`rec acc @{'exists (λ${ident x}:$T.$acc)}`

(**rec** is the binder, *acc* the bound variable and the rest is the body) over the initial content expression *@{$Px}*.

When a notation is not a simple infix operator, it may be the case that a different precedence is required for its arguments. In this case we can use the *term n X* syntax to declare that the term matched by the variable *X* needs to be parsed or pretty-printed at precedence *n*. For example, knowing that conjunction has precedence 35, by requiring *Px* to be parsed at level 19 in the notation of the existential, an expression like $\exists x.P \wedge Q$ will be parsed as $\exists x.(P \wedge Q)$, whereas it would be parsed as $(\exists x.P) \wedge Q$ if *Px* was required to be at any level greater than 35. The maximal level of precedence is 90. By asking an argument to have precedence 90, we force the argument to be always either a single identifier or to be delimited (e.g. by parentheses).

The user is invited to consult "*basics/core_notation.ma*" for more examples of declarations of complex notations.

3.4 Leibniz Equality

Even equality is a derived notion in Matita, and a particular case of an inductive type. The idea is to define it as the smallest relation containing reflexivity (that is, as the smallest reflexive relation on a given type)[6].

```
inductive eq (A:Type[0]) (x:A) : A → Prop :=
    refl: eq A x x.
```

We can associate the standard infix notation for equality via the following declarations:

```
notation > "hvbox(a break = b)"
  non associative with precedence 45 for @{ 'eq ? $a $b }.

interpretation "leibnitz's equality" 'eq t x y = (eq t x y).
```

The induction principle *eq_ind* automatically generated by the system has the following shape after removing type dependencies for clarity:

$$\forall A{:}Type[0].\forall x{:}A.\ \forall P{:}A \to Prop.\ P\ x \to \forall y{:}A.\ x{=}y \to P\ y$$

This principle is usually known as "Leibniz equality": two objects x and y are equal if they cannot be told apart, that is for any property P, $P\ x$ implies $P\ y$.

The order of the arguments in *eq_ind* may look a bit random but, as we shall see, it is motivated by the underlying structure of the inductive type. Before discussing the way *eq_ind* is generated, it is time to have an important discussion about the parameters of inductive types. If you look back at the definition of equality, you will see that the first argument x has been explicitly declared, together with A, as a formal parameter of the inductive type, while the second argument has been left implicit in the resulting type $A\to Prop$. One could wonder if this really matters, and in particular if we could use the following alternative definitions:

```
inductive eq1 (A:Type[0]) (x,y:A) : Prop :=
    refl1: eq1 A x x.
```

```
inductive eq2 (A:Type[0]): A → A → Prop :=
    refl2: ∀x.eq2 A x x.
```

The first definition is wrong. If you try to write it in Matita, you would get the following error message: "*CicUnification failure: Can't unify x* **with** *y*". The issue is that the role of parameters is really to define a family of types uniformly indexed over them. This means that we expect all occurrences of the inductive type in the type of constructors to be precisely instantiated with the input parameters, in the order they are declared. If A,x and y are parameters for *eq1*, then all occurrences of this type in the type of constructors must be of the kind *eq1 A x y* while we have *eq1 A x x*, that explains the typing error.

If you cannot express an argument as a parameter, the only alternative is to implicitly declare it in the type of the inductive type. Henceforth, when we talk

[6] The definition can only be used to equate inhabitants of small types (of type $Type[0]$). Matita does not provide a way to define a property that works uniformly at every universe. Therefore, in the rare cases when equality is needed at higher universes, the user needs to duplicate its definition or to re-declare it to work on, say, $Type[1]$. Every type that inhabits a $Type[i]$ is also recognized to be of type $Type[j]$ for every $j > i$.

about "arguments" of inductive types, we shall implicitly refer to arguments which *are not parameters* . Sometimes, people call them "right" and "left" parameters, according to their position w.r.t the colon in the type declaration. In general, it is always possible to declare everything as an argument, but it is a *very good practice* to shift as many argument as possible in parameter position. As we shall see, the induction principle generated for *eq2* is harder to understand than *eq_ind*.

The elimination rule for an inductive type T having a list of parameters $\vec{A}$ and a list of arguments $\vec{B}$ has the following shape (still, up to dependencies):

$$\forall \vec{a}:\vec{A}, P:\vec{B} \to Prop.\ C_1 \to \ldots \to C_n \to \forall \vec{x}:\vec{B}. T\ \vec{a}\ \vec{x} \to P\ \vec{x}$$

where C_i is obtained from the type T_i of the constructor c_i replacing in it each occurrence of $T\ \vec{a}\ \vec{t}$ with $P\ \vec{t}$.

For instance, *eq2* only has A as parameter, and two arguments. The corresponding elimination principle *eq2_ind* is then as follows:

$$\forall A{:}Type[0].\ \forall P{:}A \to A \to Prop.\ \forall z.\ P\ z\ z \to \forall x,y{:}A.\ x{=}y \to P\ x\ y$$

It is possible to prove that *eq2_ind* and *eq_ind* are logically equivalent (that is, they mutually imply each other), but *eq2_ind* is slighty more complex and unnatural.

3.5 Equality, convertibility, inequality

Leibniz equality is a pretty syntactic (intensional) notion: two objects are equal when they are the "same" term. There is however, an important point to understand here: the notion of sameness on terms is *convertibility*, the smallest equivalence relation containing reduction. When reduction is confluent, two terms are convertible if they have the same normal form. For this reason, not only $2 = 2$ but also $1 + 1 = 2$ since the normal form of $1 + 1$ is 2.

Having understood the notion of equality, one could easily wonder how we can prove that two objects are different. For instance, in Peano arithmetic, the fact that for any x, $0 \neq Sx$ is an independent axiom. With our inductive definition on natural numbers of Section 2, can we prove it, or are we supposed to assume such a property axiomatically?

In fact, in a logical system like the Calculus of Inductive Constructions it is possible to prove it. We shall discuss the proof here, since it is both elegant and instructive.

The crucial point is to define, by case analysis on the structure of inductive terms, a characteristic property for the different cases.

For instance, in the case of natural numbers, we could define a property *not_zero* as follows:

```
definition not_zero: nat → Prop :=
λn: nat. match n with [ O ⇒ False | (S p) ⇒ True ].
```

The possibility of defining predicates by structural recursion on terms is one of the major characteristics of the Calculus of Inductive Constructions, known as *strong elimination.*

Suppose now we want to prove the following property:

```
theorem not_eq_O_S : ∀n:nat. O = S n → False.
```

After introducing *n* and the hypothesis *H*:*O* = *S n* we are left with the goal *False*. Now, observe that also *not_zero O* is false: actually *not_zero O* *reduces* to *False* and the two terms are actually *convertible*, that is identical. So, it should be possible to *replace* *False* with *not_zero O* in the conclusion, since they are the same term.

The tactic that does the job is the *change with* tactic. The invocation of **change with** *t* checks that the current goal is convertible with *t*, and in this case *t* becomes the new current goal.

In our case, typing

```
change with (not_zero O);
```

we get the new goal *not_zero O*. But we know, by *H*, that *O*=*S n*, hence by rewriting we get the goal *not_zero (S n)* that *reduces* to *True*, whose proof is trivial (use *@I* or //).

Here is the complete proof

```
#n #H change with (not_zero O); >H // qed.
```

Using a similar technique you can always prove that different constructors of the same inductive type are distinct from each other; actually, this technique is also at the core of the *destruct* tactics of Section 2.5 in order to automatically close absurd cases.

3.6 Inversion

The only type that really needs arguments is equality. In all other cases you could conceptually get rid of them by adding, inside the type of constructors, the suitable equalities that would allow to turn arguments into parameters.

Consider for instance our *opp* relation of Section 1.6:

```
inductive opp : bank → bank → Prop :=
| east_west : opp east west
| west_east : opp west east.
```

The predicate has two arguments, and since they are mixed up in the type of constructors we cannot express them as parameters. However, we could state it in the following alternative way:

```
inductive opp1 (b1,b2: bank): Prop :=
| east_west : b1 = east → b2 = west → opp1 b1 b2
| west_east : b1 = west → b2 = east → opp1 b1 b2
```

Or also, more elegantly:

```
inductive opp2 (b1,b2: bank): Prop :=
| opposite_to_opp2 : b1 = opposite b2 → opp2 b1 b2
```

Now, suppose we know *H*:*opp x west*, where *x* is a variable of type *bank*. From this hypothesis we should be able to conclude *x*=*east*, but this is slightly less trivial than one could expect. One would be naturally tempted to proceed by case-analysis on *H*, but this would lead him nowhere: in fact it would generate the following two subgoals:

$$G_1 : \texttt{east=east}$$
$$G_2 : \texttt{west=east}$$

where the second one cannot be proved. Also induction on *x* does not help, since we would get the goals

$$G_1 : \textit{opp east west} \rightarrow \textit{east=east}$$
$$G_2 : \textit{opp west west} \rightarrow \textit{west=east}$$

The first goal is trivial, but proving that *opp west west* is absurd has about the same complexity of the original problem. In fact, the best approach consists of *generalizing* the statement to something similar to *opp_to_opposite* and then prove it as a corollary of the latter.

It is interesting to observe that we would not have the same problem with *opp1* or *opp2*. For instance, by cases analysis on *H*: *opp1 x west*, we would obtain the two subgoals

$$G_1 : \textit{x=east} \rightarrow \textit{west=west} \rightarrow \textit{x=east}$$
$$G_2 : \textit{x=west} \rightarrow \textit{west=east} \rightarrow \textit{x=east}$$

The first one is trivial, while the second one is easily closed using *destruct*.

The point is that naive pattern matching is not powerful enough to discriminate the structure of *arguments* of inductive types. To this aim, however, you may exploit an alternative tactic called *inversion*.

Suppose to have in the local context an expression *H*: *T* $\vec{t}$, where *T* is some inductive type. Then, **inversion** *H* derives for each possible constructor c_i of *T* $\vec{t}$ *all the necessary conditions* that should hold for the instance *T* $\vec{t}$ to be proved by c_i.

For instance, if the current goal is *x=east*, invoking *inversion* on the hypothesis *H*: *opp x west* would result in the following subgoals:

$$G_1 : \textit{x=east} \rightarrow \textit{west=west} \rightarrow \textit{x=east}$$
$$G_2 : \textit{x=west} \rightarrow \textit{west=east} \rightarrow \textit{x=east}$$

that are precisely the same goals generated by case analysis on *opp1*.

Remark 1. *Invoking inversion on inductive types without arguments does not make any sense, and has no practical effect.*

4. PROPOSITIONS AS TYPES

In the previous section, we introduced several logical connectives by means of inductive definitions in the sort *Prop*. Do the same constructions make any sense in *Type*[*i*]? The answer is yes! Not only do they make sense, but they are the familiar type constructors: Cartesian product, disjoint sum, empty and singleton types, and "sigma types" (disjoint unions of families of types indexed over a given base type). This is not a coincidence, but an instance of a principle known under the name of "Propositions as Types analogy" (or Curry-Howard correspondence).

We shall first discuss the constructions, and then we shall come back on the general correspondence.

4.1 Cartesian Product and Disjoint Sum

The Cartesian product of two types *A* and *B* is defined in the following way:

```
inductive Prod (A,B:Type[0]) : Type[0] :=
  pair : A → B → Prod A B
```

Observe that the definition is identical to the definition of logical conjunction, but for the fact that the sort *Prop* has been replaced by *Type*[0].

The following declarations allow us to use the canonical notations *A*×*B* for the product and ⟨*a*,*b*⟩ for the pair of the two elements *a* and *b*.

```
notation "hvbox(x break ×y)" with precedence 70 for @{ 'product $x $y}.
interpretation "Cartesian product" 'product A B = (Prod A B).

notation "hvbox(⟨term 19 a, break term 19 b⟩)"
  with precedence 90 for @{ 'pair $a $b}.
interpretation "Pair construction" 'pair x y = (pair ?? x y).
```

We can define the two projections *fst* and *snd* as a simple case analysis on the term:

```
definition fst :=λA,B.λp:A×B. match p with [pair a b ⇒ a].
definition snd :=λA,B.λp:A×B. match p with [pair a b ⇒ b].
```

As in the case of inductive propositions, Matita automatically generates elimination principles for *A*×*B*. In this case, however, it is interesting to consider the possibility that the proposition towards which we are eliminating a given pair *p*: *A*×*B* contains a copy of *p* itself. In other words, if we have *p*: *A*×*B* in the current context, it is possible that *p* also occurs in the current goal, that is that the current goal *depends* on *p*[7].

A typical example is in the proof of the so called "surjective pairing" property:

```
lemma surjective_pair: ∀A,B.∀p:A×B. p = ⟨fst ?? p, snd ?? p⟩
```

where *p* occurs in the conclusion. The proof is straightforward: we introduce *A*, *B* and *p* and proceed by case-analysis on *p*: since *p* is a pair, the only possible case is that it is of the form ⟨*a*,*b*⟩ for some *a* and *b*. At this point the goal looks like

[7]This is also possible when we are eliminating a proof *h* of a conjunction, since proofs are first class and may occur inside other types, but it is less frequent.

$$\langle a,b\rangle = \langle fst\ ??\ \langle a,b\rangle, snd\ ??\ \langle a,b\rangle\rangle$$

and the two sides of the equation are convertible. The whole proof is hence:

```
#A #B #p cases p #a #b // qed. (* #p cases p can be replaced by ‘‘*’’ *)
```

When we call `cases p` we are actually applying the dependent elimination principle for the product, so it becomes interesting to have a look at it:

$$\forall A,B.\forall P{:}A\times B \to Prop.\ (\forall a{:}A,\forall b{:}B.\ P\ \langle a,b\rangle) \to \forall x{:}A\times B.\ P\ x$$

The previous principle has a very natural backward reading: in order to prove that the property $P\ x$ holds for any x of type $A\times B$ is is enough to prove $P\ \langle a,b\rangle$ for any $a{:}A$ and $b{:}B$.

By reflecting in `Type[0]` the definition of logical disjunction we obtain the disjoint union (the sum) of two types:

```
inductive Sum (A,B:Type[0]) : Type[0] :=
  inl : A → Sum A B
| inr : B → Sum A B.

notation "hvbox(a break + b)"
  left associative with precedence 55 for @{ 'plus $a $b }.

interpretation "Disjoint union" 'plus A B = (Sum A B).
```

The two constructors are the left and right injections. The dependent elimination principle has the following shape:

$$\forall A,B.\forall P{:}A{+}B \to Prop.\ (\forall a{:}A.P\ (inl\ a)) \to (\forall b{:}B.P\ (inr\ b)) \to \forall x{:}A{+}B.\ P\ x$$

that is, in order to prove the property $P\ x$ for any $x{:}A{+}B$ it is enough to prove $P\ (inl\ a)$ and $P\ (inr\ b)$ for all $a{:}A$ and $b{:}B$.

The counterparts of *False* and *True* are, respectively, an empty type and a singleton type:

```
inductive void : Type[0] :=.
inductive unit : Type[0] := it: unit.
```

The elimination principle for void is simply

$$\forall P{:}void \to Prop.\ \forall x{:}void.\ P\ x$$

stating that any property is true for an element of type void (since we have no such element).

Similarly, the elimination principle for the unit type is:

$$\forall P{:}unit \to Prop.\ P\ it \to \forall x{:}unit.\ P\ x$$

Indeed, in order to prove that a property is true for any element of the unit type, it is enough to prove it for the unique (canonical) inhabitant *it*.

As an exercise, let us prove that all inhabitants of *unit* are equal to each other:

```
lemma eq_unit: ∀a,b:unit. a = b.
```

The idea is to proceed by case-analysis on *a* and *b*: we have only one possibility, namely *a*=*it* and *b*=*it*, hence we end up proving *it*=*it*, that is trivial. Here is the proof:

```
#a cases a #b cases b // qed. (* also: * * // qed. *).
```

It is interesting to observe that we get exactly the same behavior by directly applying *unit_ind* instead of proceeding by case-analysis. In fact, this is an alternative proof:

```
@unit_ind @unit_ind // qed.
```

4.2 Sigma Types and dependent matching

As a final example, let us consider "type" variants of the existential quantifier; in this case we have two interesting possibilities:

```
inductive Sig (A:Type[0]) (Q:A →Prop) : Type[0] :=
   Sig_intro: ∀x:A. Q x →Sig A Q.

inductive DPair (A:Type[0]) (Q:A →Type[0]) : Type[0] :=
   DPair_intro: ∀x:A. Q x →DPair A Q.
```

In the first case (traditionally called sigma type), an element of type *Sig A P* is a pair *Sig_intro a h* (we shall use the traditional pair notation $\langle a\ ,h\rangle$) where *a* is an element of type *A* and *h* is a proof that the property *P a* holds. A suggestive reading of of *Sig A P* is as the subset of *A* satisfying the property *P*, that is {*a*:*A*|*P*(*a*)}.

In the second case, an element of *DPair A B* is a *dependent* pair *DProd_intro a h* (we shall use the following notation in this case: $\ll a,h\gg$) where *a* is element of type *a* and *h* maps *a* into an element in *B a*; the intuition is to look at *DProd A B* as a (disjoint) family of sets *B a* indexed over elements *a*:*A*.

In both cases it is possible to define projections extracting the two components of the pair. Let us discuss the case of the sigma type (the other one is analogous).

Extracting the first component (the element) is easy:

```
definition Sig_fst :=λA:Type[0].λP:A→Prop.λx:Sig A P.
  match x with [Sig_intro a h ⇒ a].
```

Getting the second component is a bit trickier. The first problem is the type of the result: given a pair $\langle a,h\rangle$: *Sig A P* the type of *h* is *P a*, that is *P* applied to the first argument of the pair of which we want to extract the second component. Luckily, in a language with dependent types, it is not a problem to write such a type:

```
definition Sig_snd : ∀A,P.∀x:Sig A P.P (Sig_fst A P x) :=...
```

A subtler problem occurs when we define the body. If we just write

```
definition Sig_snd : ∀A,P.∀x:Sig A P.P (Sig_fst A P x) :=λA,P,x.
  match x with [Sig_intro a h ⇒ h].
```

the system will complain about the type of *h*. The issue is that the type of this term depends on *a*, that itself depends non-trivially on the input argument *x*. In such a situation, the type inference algorithm of Matita requires a little help: in particular the user is asked to explicitly provide the return type of the match expression, that is a map uniformly expressing the type of all branches of the case as a function of the matched expression. In our case we only have one branch and the return type is `λx.P (Sig_fst A P x)`; we declare such a type immediately after the match, introduced by the keyword "return":

```
definition Sig_snd : ∀A,P.∀x:Sig A P.P (Sig_fst A P x) :=λA,P,x.
  match x return λx.P (Sig_fst A P x) with [Sig_intro a h ⇒ h].
```

Remark 2. *We already did a systematic abuse of the arrow symbol: the expression* `A → B` *was sometimes interpreted as the* implication *between* `A` *and* `B` *and sometimes as the* function space *between* `A` *and* `B`. *Actually, in a foundational system like the Calculus of Construction, they are* the very same notion*: we only distinguish them according to the sort of the resulting expression. Similarly for the dependent product* `Πx:A.B`*: if the resulting expression is of sort* `Prop` *we shall look at it as a* universal quantification *(using the notation* `∀x:A.B`*), while if it is in some* `Type[i]` *we shall typically regard it as a* generalized Cartesian product *of a family of types* `B` *indexed over a base type* `A`.

4.3 Kolmogorov interpretation

The previous analogy between propositions and types is a consequence of a deep philosophical interpretation of the notion of proof in terms of a constructive procedure that transforms proofs of the premises into a proof of the conclusion. This is usually referred to as Kolmogorov interpretation, or Brouwer-Heyting-Kolmogorov (BHK) interpretation.

The interpretation states what is intended to be a proof of a given proposition and is specified by induction on its structure:

—a proof of `P∧ Q` is a pair `⟨a, b⟩` where `a` is a proof of `P` and `b` is a proof of `Q`;

—a proof of `P∨ Q` is a pair `⟨a, b⟩` where either `a` is `0` and `b` is a proof of `P`, or `a` is 1+ and `b` is a proof of `Q`;

—a proof of `P→ Q` is a function `f` which transforms a proof of `P` into a proof of `Q`

—a proof of `∃x:S. P x` is a pair `⟨a, b⟩` where `a` is an element of `S`, and `b` is a proof of `P a`;

—a proof of `∀x:S. P x` is a function `f` which transforms an element `a` of `S` into a proof of `P a`;

—the proposition `¬ P` is defined as `P → False`, so a proof of it is a function `f` which transforms a proof of `P` into a proof of `False`

—there is no proof of `False`.

For instance, a proof of the proposition `P → P` is a function transforming a proof of `P` into a proof of `P`: the identity function will do.

You can explicitly exploit this idea for writing proofs in Matita. Instead of declaring a lemma and proving it interactively, you may define your lemma as if it were the type of an expression, and directly proceed to inhabit it with its proof:

```
definition trivial: ∀P:Prop.P→ P :=λP,h.h.
```

It is interesting to observe that this is really the *same proof* (intensionally!) that would be produced interactively, as testified by the following fragment of code:

```
lemma trivial1: ∀P:Prop.P→ P. #P #h @h qed.
lemma same_proofs: trivial = trivial1. @refl qed.
```

Even more interestingly, we can do the opposite, namely define functions interactively. Suppose for instance that we need to define a function with the following type:

$$\forall A,B,C{:}Type[0].\ (A \to B \to C) \to A\times B \to C$$

If we just declare the type of the function followed by a fullstop, Matita will start an interactive session completely analogous to a proof session, and we can use the usual tactics to "close the goal", that is to inhabit the type.

```
definition uncurrying: ∀A,B,C:Type[0]. (A→ B→ C)→ A×B→ C.
#A #B #C #f * @f qed.
```

If you are not convinced you generated a function, or – and this is plausible – you are not so sure about the function you generated you can check your term, or test it, like in the following example (**example** is just another Matita keyword similar to **lemma** or **theorem**; examples may be referred to in proofs, but are not used by automation)

```
example uncurrying_ex: uncurrying ...plus ⟨2,3⟩ = 5. @refl qed.
```

4.4 The Curry-Howard correspondence

The philosophical ideas contained in the BHK interpretation of a proof as constructive procedure building a proof of the conclusion from proofs of the hypothesis get a precise syntactic systematization via the Curry-Howard correspondence, expressing a direct relationship between computer programs and proofs. The Curry-Howard correspondence, also known as proofs-as-programs analogy, is a generalization of a syntactic analogy between systems of formal logic and computational calculi first observed by Curry for Combinatory Logic and Hilbert-style deduction systems, and later by Howard for λ-calculus and Natural Deduction: in both cases the formation rules for well typed terms have *precisely* the same shape of the logical rules of introduction of the correspondent connective (as we already exploited in Sections 4.1 and 4.2).

So, the expression

$$M : A$$

really has a double possible reading:

(1) M is a term of type A

(2) M is a proof of the proposition A

In both cases, M is a *witness* that the object A is *inhabited*. A free variable $x : A$ is an assumption about the validity of A (we assume to have an unknown proof named x). Let us consider the cases of the introduction and elimination rule of the implication in natural deduction, that are particularly interesting:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \qquad\qquad \frac{\Gamma \vdash A \rightarrow B \qquad \Gamma \vdash A}{\Gamma \vdash B}$$

The first step is to enrich the representation by decorating propositions with explicit proof terms. In particular, propositions in the context, being assumptions, will be decorated with (pairwise distinct) free variables, while the conclusion will be decorated with a term whose free variables appear in the context.

Suppose $\Gamma, x : A \vdash M : B$: what is the expected decoration for $A \rightarrow B$? According to the Kolmogorov interpretation, M is a procedure transforming the proof $x : A$ into a proof of B; the proof of $A \rightarrow B$ is hence the function that, taken x as input, returns M, and our canonical notation for expressing such a function is $\lambda x : A.M$. Hence we get:

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x{:}A.M : A \rightarrow B}$$

that is *precisely* the typing rule for functions.

Similarly, let us suppose that $\Gamma \vdash M : A \rightarrow B$ and $\Gamma \vdash N : A$, and let us derive a natural proof decoration for the arrow elimination rule (that is just the well known *modus ponens* rule). Again, we get inspiration from Kolmogorov interpretation: a proof $M : A \rightarrow B$ is a function transforming a proof of A into a proof of B hence, since we have a proof $N : A$ in order to get a proof of B it is enough to *apply* M to the argument N:

$$\frac{\Gamma \vdash M : A \rightarrow B \qquad \Gamma \vdash N : A}{\Gamma \vdash (MN) : B}$$

But this is nothing else than the typing rule for application!

There is still a point that deserves discussion: the most interesting point of programs is that they can be executed (in a functional setting, terms can be reduced to a normal form). By the Curry-Howard correspondence, this should correspond to a normalization procedure on proofs: does this operation make any sense at the logical level? Again, the answer is yes: not only it makes sense, but it was independently investigated in the field of proof theory. A reducible expression corresponds to what is traditionally called a *cut*: a logical "detour" typically arising by an introduction rule immediately followed by an elimination rule for the same connective, as in a β-redex, where we have a direct *interaction* between an application and a λ abstraction:

$$\lambda x.MN$$

One of the main meta-theoretical results that is usually investigated on proof systems is if they enjoy the so called *cut-elimination* principle, that is if the cut-rule is *admissible*: any proof that makes use of cuts can be turned into a cut-free proof. Since cuts are redexes, a cut-free proof is a term that does not contain any redex, that is a term in normal form. Hence, the system enjoys cut-elimination if and only if the corresponding calculus is normalizing.

Cut-elimination is a major tool of proof theory, with important implications on e.g. *consistency*, *automation* and *interpolation*.

Logic	*Programming*
proposition	type
proof	program
cut	reducible expression (redex)
cut free	normal form
cut elimination	normalization
correctness verification	type checking

Fig. 2. Curry-Howard Correspondence

Let us make a final remark. If a program is a proof, then what corresponds to the act of verifying the correctness of a proof M of some proposition A? Well, M is just an expression that is supposed to have type A, so *proof verification* is nothing else than *type checking*!

This should also clarify that proof verification is a *much easier* task than proof search. While the former corresponds to type checking, the second one corresponds to the automatic synthesis of a program from a specification!

The main ideas behind the Curry-Howard correspondence are summarized in Figure 2.

4.5 Prop vs. Type

In view of the Curry-Howard analogy, the reader could wonder if there is any actual difference between the two sorts *Prop* and *Type* in a system like the Calculus of Constructions, or if it is just a matter of flavour.

In fact, there is a subtle difference concerning the type of product types over the given sorts. Consider for instance a higher order statement like $\forall P$:*Prop*.P. This is just a sentence asserting that any proposition is true, and it looks natural to look at it as an object of sort *Prop*. However, if this is the case, when we are quantifying on all propositions we are also implicitly quantifying over the proposition we are in the act of defining, that creates a strange and possibly dangerous circularity.

In mathematics, definitions of this kind, where the definition (directly or indirectly) mentions or quantifies on the entity being defined are called *impredicative*.

The opposite notion to impredicativity is predicativity, which essentially entails building stratified (or ramified) theories where quantification on lower levels results in objects of some new type.

Impredicativity can be dangerous: a well known example is Russell's "set of all sets" resulting in famous logical paradox. Consider in particular, the set U of "all sets" not containing themself as an element. Does such a set contain itself as an element? If it does then by definition it should not, and if it does not then by definition it should.

A predicative approach would consist of distinguishing e.g. between "small" sets and "large" sets, where the set of all small sets would result in a large set.

In fact, if we look at $\forall P$:*Type*`[0]`.P as a dependent product, and we identify *Type*`[0]` as a universe of (small) "sets", it would seem strange to conclude that quantifying on all (small) sets we could obtain another (small) set.

In Matita, $\forall P$:*Type*`[0]`.P has in fact type *Type*`[1]` and not *Type*`[0]`, where *Type*`[1]` is also the type of *Type*`[0]`.

So, while *Prop* is impredicative, sorts of the kind *Type*[*i*] define a potentially infinite hierarchy of predicative sorts.

The difference between predicative and impredicative sorts is in the typing rule for the product $\Pi x{:}A.B$.

If the sort of *B* is *Prop* then $\Pi x{:}A.B$: *Prop* (whatever the sort of *A* is).

If *A*:*Type*[*i*] and *B*:*Type*[*j*], then $\Pi x{:}A.B$: *Type*[*k*] where *Type*[*k*] is the smallest sort strictly larger than *Type*[*i*] and *Type*[*j*]:

$$\frac{\Gamma \vdash A : s \quad \Gamma, x : A \vdash B : \mathrm{Prop}}{\Gamma \vdash \prod x : A.B : \mathrm{Prop}} \qquad \frac{\begin{array}{c}\mathrm{Type[k]} = \mathrm{Type[i]} \sqcup \mathrm{Type[j]} \\ \Gamma \vdash A : \mathrm{Type[i]} \quad \Gamma, x : A \vdash B : \mathrm{Type[j]}\end{array}}{\Gamma \vdash \prod x : A.B : \mathrm{Type[k]}}$$

It is worth observing that in Matita, the index *i* is just a label: constraints among universes are declared by the user. The standard library (see *basics/pts.ma*) declares a few of them, with the following relations:

Type[0] < *Type*[1] < *Type*[2] < ...

The impredicativity of *Prop* is not a problem from the point of view of logical consistency, but there is a price to be paid for this: we are not allowed to eliminate a proposition (of type *Prop*) when the current conclusion is a type (of type *Type*[*i*]). In concrete terms this means that while we are allowed to build terms, types or even propositions by structural induction on terms, the only thing we can do by structural induction/case analysis on *proofs* is to build other proofs.

For instance, we know that a proof of *p*:*A*∨ *B* is either a proof of *A* or a proof of *B*, and one could be tempted to define a function that returns the boolean *true* in the first case and *false* otherwise, by performing a simple case analysis on *p*:

```
definition discriminate_to_bool :=λA,B:Prop.λp:A∨ B.
 match p with
 [ or_introl a ⇒ true
 | or_intror b ⇒ false
 ].
```

If you type the previous definition, Matita will complain with the following error: "*TypeChecker failure*: *Sort elimination not allowed*: *Prop towards Type*[0]". Even returning the two *propositions* *True* and *False* instead of two booleans would not work:

```
definition discriminate_to_Prop :=λA,B:Prop.λp:A∨ B.
 match p with
 [ or_introl a ⇒ True
 | or_intror b ⇒ False
 ].
```

The error message is the same as before: in both cases the sort of the branches (the right hand sides of the pattern matching construct) is *Type*[0]. The only thing we can do is return other proofs, like in the following example:

```
definition or_elim :=λA,B,C:Prop.λp:A∨ B.λf:A→ C.λg:B→ C.
 match p with
```

```
[ or_introl a ⇒ f a
| or_intror b ⇒ g b
].
```

Exercise: repeat the previous examples in interactive mode, by elimination on the hypothesis `p:A∨ B`.

5. MORE DATA TYPES

In this section we shall consider more examples of datatypes, starting from quite traditional ones, like option types and lists, to conclude with less common structures, like finite sets, or vectors with a specified length.

5.1 Option Type

Matita can only define total functions. However, we are not always in a condition where we can return a meaningful value: for instance, working on natural numbers, the predecessor of `0` is undefined. In these situations, we may either return a default value (usually, `pred 0 = 0`), or decide to return an *option type* as a result. An option type is a polymorphic type that represents encapsulation of an optional value. It consists of either an empty constructor (traditionally called `None`), or a constructor encapsulating the original data type `A` (written `Some A`):

```
inductive option (A:Type[0]) : Type[0] :=
   None : option A
 | Some : A → option A.
```

The type `option A` is isomorphic to the disjoint sum between `unit` and `A`. The two bijections are simply defined as follows:

```
definition option_to_sum :=λA.λx:option A.
  match x with
  [ None ⇒ inl ?? it
  | Some a ⇒ inr ?? a ].
```

```
definition sum_to_option :=λA.λx:unit+A.
  match x with
  [ inl a ⇒ None A
  | inr a ⇒ Some A a ].
```

The fact that going from the option type to the sum and back again we get the original element follows from a straightforward case analysis:

```
lemma option_bij1: ∀A,x. sum_to_option A (option_to_sum A x) = x.
#A * // qed.
```

The other direction is just slightly more complex, since we need to exploit the fact that the unit type contains a single element: we could use lemma `eq_unit` or proceed by case-analysis on the unit element.

```
lemma option_bij2: ∀A,x. option_to_sum A (sum_to_option A x) = x.
#A * // * // qed.
```

We shall see more examples of functions involving the option type in the next section.

5.2 Lists

Options, products and sums are simple examples of polymorphic types, that is, types that depend in a uniform and effective way on other types. This form of polymorphism (sometimes called *parametric* polymporphism to distinguish it from *ad hoc* polymorphism, also known as *overloading*) is a major feature of modern functional programming languages: in many interesting cases, it allows us to write *generic* functions operating on inputs without depending on their type. We illustrate this with a typical polymorphic type, lists. For instance, appending two lists is an operation that is essentially independent of the type *A* of the elements in the list, and we would like to write a *single* append function working parametrically on *A*.

The first step is to define a type for lists polimorphic on the type *A* of its elements:

```
inductive list (A:Type[0]) : Type[0] :=
  | nil: list A
  | cons: A → list A → list A.
```

The definition should be clear: a list is either empty (*nil*) or is obtained by prefixing (*cons*) an element of type *A* to a given list. In other words, the type of lists is the smallest inductive type generated by the two constructors *nil* and *cons*.

The first element of a list is called its *head*. If the list is empty the head is undefined: as discussed in the previous section, we should either return a “default” element, or decide to return an option type. We have an additional complication in this case, since the “default” element should have type *A* (as the head), and we know nothing about *A* (it could even be an empty type!). We have no way to guess a canonical element, and the only possibility (along this approach) is to take it as input. These are the two options:

```
definition hd :=λA.λl: list A.λd:A.
  match l with [ nil ⇒ d | cons a _ ⇒ a].

definition option_hd :=
  λA.λl:list A. match l with
  [ nil ⇒ None ?
  | cons a _ ⇒ Some ? a ].
```

What remains of a list after removing its head is called the *tail* of the list. Again, the tail of an empty list is undefined, but in this case the result must be a list, and we have a canonical inhabitant of lists, that is the empty list. So, it is natural to extend the definition of tail saying that the tail of nil is nil (that is very similar to saying that the predecessor of 0 is 0):

```
definition tail :=λA.λl: list A.
  match l with [ nil ⇒ [] | cons hd tl ⇒ tl].
```

Using destruct, it is easy to prove that *cons* is injective in both arguments.

```
lemma cons_injective_l : ∀A.∀a1,a2:A.∀l1,l2.a1::l1 = a2::l2 → a1 = a2.
#A #a1 #a2 #l1 #l2 #Heq destruct // qed.

lemma cons_injective_r : ∀A.∀a1,a2:A.∀l1,l2.a1::l1 = a2::l2 → l1 = l2.
#A #a1 #a2 #l1 #l2 #Heq destruct // qed.
```

The append function is defined by recursion on the first list:

```
let rec append A (l1: list A) l2 on l1 :=
  match l1 with
  [ nil ⇒ l2
  | cons hd tl ⇒ hd :: append A tl l2 ].
```

We leave it to the reader to prove that *append* is associative, and that *nil* is a neutral element for it.

Before discussing more operations over lists, it is convenient to introduce a bit of notation; we shall write a::l for *cons a l*, [a_1;a_2,...,a_n] for the list composed by the elements a_1,a_2,...,a_n and l_1@l_2 for the concatenation of l_1 and l_2. This can be obtained by means of the following declarations:

```
notation "hvbox(hd break :: tl)"
  right associative with precedence 47 for @{'cons $hd $tl}.

notation "[ list0 term 19 x sep ; ]"
  non associative with precedence 90
  for ${fold right @'nil rec acc @{'cons $x $acc}}.

notation "hvbox(l1 break @ l2)"
  right associative with precedence 47 for @{'append $l1 $l2}.

interpretation "nil" 'nil = (nil ?).
interpretation "cons" 'cons hd tl = (cons ? hd tl).
interpretation "append" 'append l1 l2 = (append ? l1 l2).
```

Note that [] is an alternative notation for the empty list, and [a] is a list containing a singleton element a.

We conclude this section discussing another important operation over lists, namely the reverse function, returning a list with the same elements as the original list but in reverse order.

One could define the reverse operation as follows:

```
let rec rev A l1 on l1 :=
  match l1 with
  [ nil ⇒ nil A
  | cons a tl ⇒ rev A tl @ [a]
  ].
```

However, this implementation is sligtly inefficient, since it has a quadratic complexity. A better solution consists of exploiting an accumulator, passing through the following auxilary function:

```
let rec rev_append S (l1,l2:list S) on l1 :=
  match l1 with
  [ nil ⇒ l2
  | cons a tl ⇒ rev_append S tl (a::l2)
  ].

definition reverse :=λS.λl.rev_append S l [].
```

Exercise Prove the following results:

```
lemma reverse_cons : ∀S,a,l. reverse S (a::l) = (reverse S l)@[a].

lemma reverse_append: ∀S,l1,l2.
  reverse S (l1 @ l2) = (reverse S l2)@(reverse S l1).

lemma reverse_reverse : ∀S,l. reverse S (reverse S l) = l.
```

For a solution, look into the file *basics/lists/list.ma* in the standard library.

5.3 List iterators

In general, an iterator for some data type is an object that enables us to traverse its structure performing a given computation. There is a canonical set of iterators on lists deriving from programming experience. In this section we shall review a few of them, proving some of their properties.

A first, famous iterator is the map function, that applies a function f to each element of a list $[a_1,\ldots,a_n]$, building the list $[f\ a_1;\ldots;f\ a_n]$.

```
let rec map (A,B:Type[0]) (f: A →B) (l: list A) on l: list B :=
 match l with
 [ nil ⇒ nil ?
 | cons x tl ⇒ f x :: (map A B f tl)
 ].
```

The *map* function distributes over *append*, as can be simply proved by induction on the first list:

```
lemma map_append : ∀A,B,f,l1,l2.
  (map A B f l1) @ (map A B f l2) = map A B f (l1@l2).
#A #B #f #l1 elim l1
  [ #l2 @refl | #h #t #IH #l2 normalize // ] qed.
```

The most important iterator is traditionally called *fold*; we shall only consider the so called fold-right variant, that takes as input a function $f{:}A \to B \to B$, a list $[a_1;\ldots;a_n]$ of elements of type A, a base element $b{:}B$ and returns the value $f\ a_1\ (f\ a_2\ (\ldots\ (f\ a_n\ b)\ \ldots))$.

```
let rec foldr (A,B:Type[0]) (f:A →B →B) (b:B) (l:list A) on l :B :=
 match l with [ nil ⇒ b | cons a l ⇒ f a (foldr A B f b l)].
```

Many other interesting functions can be defined in terms of *foldr*. A first interesting example is the filter function, taking as input a boolean test p and a list l and returning the sublist of all elements of l satisfying the test.

```
definition filter :=
  λT.λp:T → bool.
  foldr T (list T) (λx,l0.if p x then x::l0 else l0) (nil T).
```

As another example, we can generalize the map function to an operation taking as input a binary function $f{:}A \to B \to C$, two lists $[a_1;\ldots;a_n]$ and $[b_1;\ldots;b_m]$ and returning the list $[f\ a_1\ b_1;\ldots;f\ a_n\ b_1;\ldots;f\ a_1\ b_m;\ f\ a_n\ b_m]$

```
definition compose :=λA,B,C.λf:A→ B→ C.λl1,l2.
  foldr ?? (λi,acc.(map ?? (f i) l2)@acc) [ ] l1.
```

The folded function `λi,acc.(map ?? (f i) l2)@acc` takes as input an element `i` and an accumulator, and adds to the accumulator the values obtained by mapping the partial instantiation `f i` on the elements of `l2`. This function is iterated over all elements of the first list `l1`, starting with an empty accumulator.

5.4 Naive Set Theory

Given a "universe" `U` (an arbitrary type `U:Type[i]` for some `i`), a practical way to deal with subsets of `U` is simply to identify them with their characteristic property, i.e. as functions of type `U → Prop`.

For instance, the empty set is defined by the `False` predicate; a singleton set `{x}` is defined by the property that its elements are equal to `x`.

```
definition empty_set :=λU:Type[0].λu:U.False.
definition singleton :=λU.λx,u:U.x=u.
```

The membership relation is trivial: an element `x` is in the set (defined by the property) `P` if and only if it enjoys the property:

```
definition member :=λU:Type[0].λu:U.P→ Prop:U.P u.
```

The operations of union, intersection, complementation and difference are defined in a straightforward way, in terms of logical operations:

```
definition union :=λU:Type[0].λP,Q:U → Prop.λa.P a ∨ Q a.

definition intersection :=λU:Type[0].λP,Q:U → Prop.λa..P a ∧ Q a.

definition complement :=λU:Type[0].λA:U → Prop.λw.¬  A w.

definition difference :=λU:Type[0].λA,B:U → Prop.λw.A w ∧ ¬ B w.
```

More interesting are the notions of subset and equality. Given two sets `P` and `Q`, `P` is a subset of `Q` when any element `u` in `P` is also in `Q`, that is when `P u` implies `Q u`.

```
definition subset: ∀U:Type[0].∀P,Q:U→ Prop.Prop :=λU,P,Q.∀u:U.(P u → Q u).
```

Then, two sets `P` and `Q` are equal when `P ⊆ Q` and `Q ⊆ P`, or equivalently when for any element `u`, `P u ↔ Q u`.

```
definition eqP :=λU:Type[0].λP,Q:U → Prop.∀u:U.P u ↔Q u.
```

We shall use the infix notation $\doteq$ for the previous notion of equality. It is important to observe that the `eqP` is different from Leibniz equality of section 3.4. As we already observed, Leibniz equality is a pretty syntactic (or intensional) notion, that is a notion concerning the *connotation* of an object and not its *denotation* (the shape assumed by the object, and not the information it is supposed to convey). Intensionality stands in contrast with *extensionality*, referring to principles that judge objects to be equal if they enjoy *a given subset* of external, observable properties (e.g. the property of having the same elements). For instance given two sets A and B we can prove that $A \cup B \doteq B \cup A$, since they have the same elements, but there is no way to prove $A \cup B = B \cup A$.

The main practical consequence is that, while we can always exploit a Leibniz equality between two terms t_1 and t_2 for rewriting one into the other (in fact, this is the *essence* of Leibniz equality), we cannot do the same for an extensional equality (we could only rewrite inside propositions "compatible" with our external observation of the objects).

5.5 Sets with decidable equality

We say that a property $P:A \to Prop$ over a datatype A is *decidable* when we have an effective way of assessing the validity of $P\ a$ for any $a:A$. As a consequence of Gödel's incompleteness theorem, not every predicate is decidable: for instance, extensional equality of sets is not decidable, in general.

Decidability can be expressed in several possible ways. A convenient one is to state it in terms of the computability of the characteristic function of the predicate P, that is in terms of the existence of a function $Pb: A \to bool$ such that

$$P\ a \leftrightarrow Pb\ a = true$$

Decidability is an important issue, and since equality is an essential predicate, it is always interesting to try to understand when a given notion of equality is decidable or not.

In particular, Leibniz equality on inductively generated datastructures is often decidable, since we can simply write a decision algorithm by structural induction on the terms. For instance we can define characteristic functions for booleans and natural numbers in the following way:

```
definition beqb :=λb1,b2.
  match b1 with [ true ⇒ b2 | false ⇒ notb b2].

let rec neqb n m :=
match n with
  [ O ⇒ match m with [ O ⇒ true | S q ⇒ false]
  | S p ⇒ match m with [ O ⇒ false | S q ⇒ neqb p q]
  ].
```

It is so important to know if Leibniz equality for a given type is decidable that we often pack this information together into a single algebraic data structure called *DeqSet*:

```
record DeqSet : Type[1] :=
 { carr :>Type[0]
 ; eqb: carr → carr → bool
 ; eqb_true: ∀x,y. (eqb x y = true) ↔(x = y)}.
```

A *DeqSet* is simply a record composed by a carrier type *carr*, a boolean function eqb: $carr \to carr \to carr$ representing the decision algorithm, and a *proof* *eqb_true* that the decision algorithm is correct.

The definition of a *DeqSet* is an instance of a mathematical structure in algebra, that are often denoted simply by their carriers. For example, if G is a group, we write $x \in G$ to say that x belongs to the carrier of G. The same abuse of notation is achieved in Matita via *coercions*. A coercion from a type T_1 to a type T_2 is a function that is silently inserted around an expression $e_1 : T_1$ to turn it into an expression of type T_2. The :> symbol in the definition of *DeqSet* declares the projection *carr* as a coercion from a DeqSet to its carrier type. The reader may consul the user manual of Matita for the general syntax to declare coercions.

We use the infix notation "==" for the decidable equality *eqb* of the carrier. The notations \P *H* and \b *H* are used, respectively, to convert a boolean equality (*x* == *y*) = *true* into a propositional equality *x* = *y* and vice-versa, and are converted to instances of *eqb_true* (of course, *x* and *y* must belong to a *DeqSet* in order for this notation to work).

```
notation "a == b" non associative with precedence 45 for @{ 'eqb $a $b }.
interpretation "eqb" 'eqb a b = (eqb ? a b).

notation "\P H" non associative with precedence 90
  for @{(proj1 ...(eqb_true ???) $H)}.

notation "\b H" non associative with precedence 90
  for @{(proj2 ...(eqb_true ???) $H)}.
```

Suppose we proved the following facts (do it as an exercise)

```
lemma beqb_true_to_eq: ∀b1,b2. beqb b1 b2 = true ↔b1 = b2.
lemma neqb_true_to_eq: ∀n,m:nat. eqb n m = true →n = m.
```

Then, we can build the following records:

```
definition DeqBool :=mk_DeqSet bool beqb beqb_true_to_eq.
definition DeqNat :=mk_DeqSet nat eqb eqbnat_true.
```

Note that, since we declared a coercion from the *DeqSet* to its carrier, the expression *O*:*DeqNat* is well typed, and it is understood by the system as *O*: *carr DeqNat*.

5.6 Unification hints

Now, suppose we need to write an expression of the following kind:

```
b == false
```

that, after removing the notation, is equivalent to

```
eqb ? b false
```

The system knows that *false* is a boolean, so in order to accept the expression, it should *figure out* some *DeqSet* having *bool* as carrier. This is not a trivial operation: Matita should either try to synthesize it (which is a complex operation known as *narrowing*) or look into its database of results for a possible solution.

In this situation, we may suggest the intended solution in Matita, which uses the mechanism of unification hints [6, 19]. The concrete syntax of unification hints is a bit involved: we strongly recommend that the user include the file *hints_declaration.ma*, allowing us to write hints in a more convenient and readable way.

```
include "hints_declaration.ma".
```

The following declaration uses the aforementioned mechanism to *hint* that a solution of the equation `bool = carr X` is `X = DeqBool`.

```
unification hint 0 ≔ ;
    X ≟ DeqBool
(* ---------------------------------------- *) ⊢
    bool ≡ carr X.
```

Using the previous notation (we suggest the reader to cut and paste it from previous hints) the hint is expressed as an inference rule. The conclusion of the rule is the unification problem that we intend to solve, containing one or more variables $X_1, \ldots, X_b$. The premises of the rule are the solutions we are suggesting Matita. In general, unification hints should only be used when there exists just one "intended" (canonical) solution for the given equation. When you declare a unification hint, Matita verifies its correctness by instantiating the unification problem with the hinted solution, and checking the convertibility between the two sides of the equation.

```
example exhint: ∀b:bool. (b == false) = true → b = false.
#b #H @(\P H).
qed.
```

In a similar way,

```
unification hint 0 ≔ b1,b2:bool;
    X ≟ mk_DeqSet bool beqb beqb_true
(* ---------------------------------------- *) ⊢
    beqb b1 b2 ≡ eqb X b1 b2.
```

5.7 Prop vs. bool

It is probably time to have a discussion about `Prop` and `bool`, and their different roles in the Calculus of Inductive Constructions.

Inhabitants of the sort `Prop` are logical propositions. In turn, logical propositions `P:Prop` can be inhabitated by their proofs `H:P`. Since, in general, the validity of a property `P` is not decidable, the role of the proof is to provide a witness of the correctness of `P` that the system can automatically check.

On the other hand, `bool` is just an inductive datatype with two constructors true and false: these are terms, not types, and cannot be inhabited. Logical connectives on bool are computable functions, defined by their truth tables, using case analysis.

`Prop` and `bool` are, in a sense, complementary: the former is more suited to abstract logical reasoning, while the latter allows, in some situations, for brute-force proving by evaluation. Suppose for instance that we wish to prove that the `2 ≤ 3!`. Starting from the definition of the factorial function and properties of the less or equal relation, the previous proof could take several steps. Suppose however we proved the decidability of ≤, that is the existence of a boolean function `leb` reflecting ≤ in the sense that

$$n \leq m \leftrightarrow \; leb\; n\; m = true$$

Then, we could reduce the verification of `2 ≤ 3!` to that of of `leb 2 3! = true` that just requires a mere computation!

5.8 Finite Sets

A finite set is a record consisting of a DeqSet *A*, a list of elements of type *A* enumerating all the elements of the set, and the proofs that the enumeration does not contain repetitions and is complete (*memb* is the membership operation on lists, defined in the obvious way, and *uniqueb* is the function that returns *true* if its third argument is an enumeration without repetitions).

```
record FinSet : Type[1] :=
{ FinSetcarr:> DeqSet
; enum: list FinSetcarr
; enum_unique: uniqueb FinSetcarr enum = true
; enum_complete : ∀x:FinSetcarr. memb ? x enum = true}.
```

The standard library of Matita provides many operations for building new *FinSet*s by composing existing ones: for example, if *A* and *B* have type *FinSet*, then *option A*, *A*×*B*, *A*+*B* are finite sets too. In all these cases, unification hints are used to suggest the *intended* finite set structure associated with them, that makes their use quite transparent to the user.

A particularly interesting case is that of the arrow type *A* → *B*. We may define the graph of *f*:*A* → *B*, as the set (sigma type) of all pairs ⟨*a*,*b*⟩ such that `f a = b`.

```
definition graph_of :=λA,B.λf:A→B. Σp:A×B.f (\fst p) = \snd p.
```

When the equality on *B* is decidable, we can effectively enumerate all elements of the graph by simply filtering from pairs in *A*×*B* those satisfying the test `λp.f (\fst p) == \snd p`, as expressed by the following definition:

```
definition graph_enum :=λA,B:FinSet.λf:A→B.
  filter ? (λp.f (\fst p) == \snd p) (enum (FinProd A B)).
```

The proofs that this enumeration does not contain repetitions and is complete are straightforward.

5.9 Vectors

A vector of length *n* of elements of type *A* is simply defined in Matita as a record composed by a list of elements of type *A*, plus a proof that the list has the expected length. Vectors are a paradigmatic example of *dependent* type, that is of a type whose definition depends on a term.

```
record Vector (A:Type[0]) (n:nat): Type[0] :=
{ vec :>list A
; len: length ? vec = n }.
```

Given a list *l* we may trivially turn it into a vector of length `|l|`; we just need to prove that `|l|=|l|`, which follows from the reflexivity of equality.

```
lemma Vector_of_list :=λA,l.mk_Vector A (|l|) l (refl ??).
```

Most functions operating on lists can be naturally extended to vectors: interesting cases are *vec_append*, concatenating vectors, and *vec_map*, mapping a function *f* on all the elements of the input vector and returning a vector of the same length as the input one.

Since a vector is automatically coerced, if needed, to the list of its elements, we may simply use typical functions over lists (such as *nth*) to extract/filter specific components.

The function *change_vec A n v a i* replaces the content of the vector *v* at position *i* with the value *a*.

The most frequent operation on vectors for our purposes is comparison. In this case, we have essentially two possible approaches: we may proceed component-wise, using the following lemma

```
lemma eq_vec: ∀A,n.∀v1,v2:Vector A n.∀d.
  (∀i. i < n → nth i A v1 d = nth i A v2 d) → v1 = v2.
```

or alternatively we may manipulate vectors by exploiting commutativity or idempotency of *change_vec* and its interplay with other operations.

5.10 Dependent matching

We are now going to present the full syntax of the pattern matching construct, as well as its typing rule, by means of simple examples concerning a type of *strictly balanced binary trees*. In standard textbooks on data structures, a binary tree is balanced when the heights of the left and right subtrees of every node differ by at most 1; our definition is *strictly* balanced because we require the subtrees of every node to have *exactly* the same height.

Concretely, we define the inductive type *SBBT n* of strictly balanced binary trees of height *n* that store natural numbers in their leafs. The constructor *leaf* has as input the natural number stored in the leaf, and builds ans SBTT of zero height. The constructor *node* combines two SBBTs t_1, t_2 of the same height *m* into a new SBBT of height *S m*. The height cannot be expressed as a parameter because different subtrees have different heights, whereas parameters must remain constant.

```
inductive SBBT : nat → Type[0] :=
| leaf : nat → SBBT O
| node : ∀n.SBBT n → SBBT n → SBBT (S n).
```

Suppose we want to write a function *subtree_left* which, given a SBBT, returns its left subtree, assuming it exists. A possible implementation uses, as the output type of the function, an optional value, which allows us to return a *None* when the input is a leaf:

```
definition subtree_left :
  ∀n.∀t:SBBT n.option (SBBT (pred n))
:=λn,t.match t with
  [ leaf k ⇒ None ?
  | node m t1 t2 ⇒ Some ? t1 ].
```

This operationally correct implementation is rejected by the system, which is unable to synthesize the return type of the match. To overcome this problem, as we did in Section 4.2, we need to add more type annotations by means of the **return** clause to the match. Since the matched type depends on an argument (the height of the tree), which changes depending on the branch of the match, the function expressing the return type must depend not only on the matched term, but also on the index *n* of its type *SBBT n*. For the *subtree_left* function, we need to specify that the output type is an optional SBBT whose height is the predecessor of that of the input tree:

```
definition subtree_left :
  ∀n.∀t:SBBT n.option (SBBT (pred n))
:=λn,t.match t return (λn0,t0.option (SBBT (pred n0))) with
  [ leaf k ⇒ None ?
  | node m t1 t2 ⇒ Some ? t1 ].
```

To typecheck the match, Matita will ensure that the type of each branch corresponds to the (properly instantiated) return type. In particular:

—in the first branch the type of *leaf k* is *SBBT O*, thus Matita verifies that *None ?* has type

$$(\lambda n_0, t_0.option(SBBT(pred n_0)))O(leaf k)$$

—in the second branch, the type of *node m t1 t2* is *SBBT (S m)*, thus Matita verifies that *Some ? t1* has type

$$(\lambda n_0, t_0.option:(SBBT(pred n_0)))(Sm)(node m t_1 t_2)$$

Since both checks succeed, the type of the match will be the return type instantiated on the matched term:

$$(\lambda n_0, t_0.option:(SBBT(pred n_0)))nt$$

which reduces to *option (SBBT (pred n))*, that is the type declared for *subtree_left*.

The typing rule for dependent matching is one of the most complex rules of the type theory implemented in Matita. A formal description of the rules can be found in [7].

5.11 A heterogeneous notion of equality

While the *SBBT n* type contains strictly balanced binary trees of a given height *n*, we might be interested in stating properties of SBBTs *regardless* of their height. In set theoretical terms, this is achieved by stating the properties on the (disjoint) union indexed over *n* of the family of sets *SBBT n*.

In type theory, this indexed disjoint union corresponds to the dependent pair type *DPair* of Section 4.2: the "unified" type of SBBTs

USBBT := DPair nat SBBT

contains all the pairs ≪*k*, *t*≫ where *k* is a natural number and *t* has type *SBBT k*.

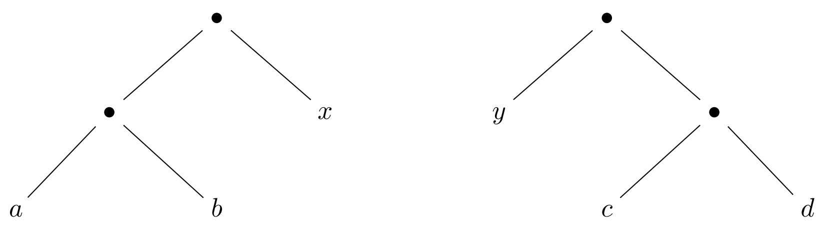
USBBT enables us to consider properties involving balanced trees of unknown height. This means, of course, that we can immediately express equalities or inequalities about any pair of USBBTs. Consider for instance the following context:

$$\begin{aligned} m,n &: nat \\ a,b &: SBBT\ m \\ c,d &: SBBT\ n \\ x &: SBBT\ (S\ m) \\ y &: SBBT\ (S\ n) \end{aligned}$$

Let us define two terms of type *USBBT*:

$$\begin{aligned} t_1 &:= \ll S\ (S\ m), node\ (S\ m)\ (node\ m\ a\ b)\ x \gg \\ t_2 &:= \ll S\ (S\ n), node\ (S\ n)\ y\ (node\ n\ c\ d) \gg \end{aligned}$$

or, graphically:



(t_1) (t_2)

Since t_1 and t_2 have the same type *USBTT*, an equation involving them is well typed, even though m and n, *a priori*, could be different. Actually, since t_1 and t_2 are dependent pairs and *DPair* is an inductive type, we expect to be able to derive, from t_1=t_2 that the two pairs are component-wise equal:

$$\begin{aligned} e_1 &: \quad S\ (S\ m) = S\ (S\ n) \\ e_2 &: \quad node\ (S\ m)\ (node\ m\ a\ b)\ x = node\ (S\ n)\ y\ (node\ n\ c\ d) \end{aligned}$$

Even though by e_1 we know that m and n are equal, e_2 is still ill-typed; for e_2 to typecheck, the knowledge that its two sides have the same type must be formalized, for instance by rewriting the left-hand side by means of e_1, using the *eq_ind* principle of Section 3.4:

$$e_2:\ eq\_ind\ ???\ (node\ (S\ m)\ (node\ m\ a\ b)\ x)\ ?\ e_1 = node\ (S\ n)\ y\ (node\ n\ c\ d)$$

eq_ind changes the type of the left-hand side from $SBBT\ (S\ (S\ m))$ to $SBBT\ (S\ (S\ n))$ and makes it possible to equate it to the right-hand side.

Working with explicit rewritings is cumbersome and becomes essentially infeasible as the number of rewriting steps grows. To directly express the equality of terms whose actual types differ (but may be made equal by means of rewriting), we have to abandon Leibniz equality and switch to a *heterogeneous equality*.

In Matita, heterogeneous equality is defined as an inductive type which, likely its Leibniz counterpart, is only inhabited by reflexivity; the difference between the two is that heterogeneous equality has an additional argument allowing the type of the right-hand side to be different from that of the left-hand side. For a concrete proof of equality built by reflexivity, the two types (and the two terms) have to be the same, but unlike with Leibniz equality, merely *stating* an equality involving two different types is still well-typed. This is the definition of heterogeneous equality, taken from the library file *basics/jmeq.ma*:

```
inductive JMeq (A:Type[0]) (x:A) : ∀B:Type[0].B → Prop :=
   refl_jmeq: JMeq A x A x.
```

For a heterogeneous equality *JMeq A a B b* equating $a{:}A$ and $b{:}B$ we use the notation $a \simeq b$. Like every inductive type, heterogeneous equality provides an induction principle: its shape *looks* quite similar to that of its Leibniz counterpart:

$$\begin{array}{l} \mathit{JMeq\_ind}\colon\ \forall A{:}\mathit{Type}[0].\forall x{:}A.\ \forall P{:}\forall T{:}\mathit{Type}[0].T \to \mathit{Prop}.\ P\ A\ x \to \\ \qquad\qquad \forall B{:}\mathit{Type}[0].\forall y{:}B.\ x \simeq y \to P\ B\ y \end{array}$$

The difference between *JMeq_ind* and *eq_ind* is analogous to that in the definition of the two equalities: in *JMeq_ind*, the scheme *P* is abstracted not only over the term to be rewritten, but also over its type, which is also going to be rewritten. This is not as nice as it sounds, because not every predicate can be made generic over the type of its argument: when this is not possible (in practice: most of the time), all attempts to use *JMeq_ind* to perform rewriting will fail.

While the above discussion may sound abstract, a very concrete example is that the property

$$\mathit{jmeq\_to\_eq}\ :\ \forall T{:}\mathit{Type}[0].\forall x,y{:}T.\, x \simeq y \to x = y$$

is *not* an immediate consequence of *JMeq_ind* – in particular, depending on certain characteristics of the type theory in use, it may even be not provable. In Matita, it is proved by exploiting *proof irrelevance* (which we will briefly discuss in Section 9.3) and declared as a coercion, allowing heterogeneous equalities where both sides have the same type to be used in tactics in place of Leibniz equalities.

Heterogeneous equality is integrated in Matita: after the file *basics/jmeq.ma* has been included, the system will create heterogeneous versions of discrimination and inversion principles for all the inductive types subsequently defined. In particular, returning to our example, if we know by hypothesis that $t_1 \simeq t_2$ and we want to prove that

$$x \simeq \mathit{node}\ n\ c\ d \wedge y \simeq \mathit{node}\ m\ a\ b$$

it is sufficient to **destruct** the hypothesis. The goal will become

$$\mathit{node}\ n\ c\ d \simeq \mathit{node}\ n\ c\ d \wedge \mathit{node}\ n\ a\ b \simeq \mathit{node}\ n\ a\ b$$

which is easily closed by reflexivity (notice that **destruct** has also performed unification in the context, changing the type of *a* and *b* to from *SBBT (S m)* to *SBBT (S n)*).

6. A FORMALIZATION EXAMPLE: REGULAR EXPRESSIONS AND DFA

As a first, non-trivial application of the notions we have been studying so far, we shall investigate in this section some basic results on regular expressions and deterministic finite automata. In particular, we shall provide a formal construction of the deterministic finite automaton associated with a given regular expression, and a fully verified bisimilarity algorithm to check regular expression equivalence.

Our approach is based on the notion of *pointed regular expression* (pre), introduced in [11, 1]. A pointed regular expression for e is just a regular expression e internally labelled with some additional points. Intuitively, points mark the positions inside the regular expression which have been reached after reading some prefix of the input string, or better, the positions where the processing of the remaining string has to be started. Each pointed expression for e represents a state of the *deterministic* automaton associated with e; since we obviously have only a finite number of possible labellings, the number of states of the automaton is finite.

Pointed regular expressions offer an appealing alternative to Brzozowski's derivatives, avoiding their weakest point, namely the fact of being forced to quotient derivatives with respect to a suitable notion of equivalence in order to get a finite number of states (that is not essential for recognizing strings, but is crucial for comparing regular expressions).

6.1 Words and Languages

An *alphabet* is an arbitrary set of elements, equipped with a decidable equality (a *DeqSet*, see Section 5.5).

A string (or word) over the alphabet S is just an element of `list S`. The empty string ϵ is then another notation for the empty list `[]`.

A language L (over an alphabet S) is a set of strings in `list S`. As described in Section 5.4, a traditional way to encode subsets of a given set U in type theory is by means of predicates over a type U, which are elements of $U \to Prop$. A language over an alphabet S is hence an element of `list S` $\to Prop$.

Languages inherit all the basic operations for sets, namely union, intersection, complementation, difference, and so on. In addition, we may define some new operations induced by string concatenation, and in particular the *concatenation* $A \cdot B$ of two languages A and B, the *Kleene's star* A^* of A and the *derivative* of a language A w.r.t. a given character a, which is the set of all strings w such that $aw \in A$:

```
definition cat :=λS,A,B.λw:word S.
  ∃w1,w2.w1 @ w2 = w ∧ A w1 ∧ B w2.

definition star :=λS,A,λw:word S.
  ∃lw.flatten S lw = w ∧ list_forall S lw A.

definition deriv :=λS,A,a,w. A (a::w).
```

In the definition of star, `flatten` and `list_forall` are standard functions over lists, mapping $[l_1,\ldots,l_n]$ to $l_1 @ l_2 \ldots @ l_n$ and $[w_1, w_2, \ldots, w_n]$ to $(A\ w_1) \wedge (A\ w_2) \ldots \wedge (A\ w_n)$.

Two languages are equal if they are equal as sets (see the definition of `eqP`, in Section 5.4), that is, if they contain the same words.

The main equations between languages that we shall need for the purposes of this tutorial (in addition to the set theoretic ones, and those expressing extensionality of operations) are listed below; the simple proofs are omitted.

```
lemma epsilon_cat_r: ∀S.∀A:word S →Prop.
     A · {ϵ} ≐ A.
lemma epsilon_cat_l: ∀S.∀A:word S →Prop.
    {ϵ} · A ≐ A.
lemma distr_cat_r: ∀S.∀A,B,C:word S →Prop.
    (A ∪ B) · C ≐ A · C ∪ B · C.
lemma deriv_union: ∀S,A,B,a.
    deriv S (A ∪B) a ≐ (deriv S A a) ∪(deriv S B a).
lemma deriv_cat: ∀S,A,B,a.
    ¬ A ϵ → deriv S (A·B) a ≐ (deriv S A a) · B.
lemma star_fix_eps : ∀S.∀A:word S →Prop.
    A* ≐ (A - {ϵ}) · A* ∪ {ϵ}.
```

6.2 Regular Expressions

The type *re* of regular expressions over an alphabet *S* is the smallest collection of objects generated by the following constructors:

```
inductive re (S: DeqSet) : Type[0] :=
   z: re S                (* empty *)
 | e: re S                (* epsilon *)
 | s: S →re S             (* symbol *)
 | c: re S →re S →re S (* concatenation *)
 | o: re S →re S →re S (* plus *)
 | k: re S →re S.         (* kleene's star *)
```

We skip the obvious notational command that allow us to use the traditional notation for regular expressions, namely $\emptyset, \epsilon, a, e_1 \cdot e_2, e_1 + e_2, e^*$ where *a* ranges over the alphabet *S*.

The language *sem r* (notation: $[\![r]\!]$) associated with the regular expression *r* is defined by the following function:

```
let rec sem (S : DeqSet) (r : re S) on r : word S →Prop :=
  match r with
  [ z ⇒∅
  | e ⇒{ϵ}
  | s x ⇒{[x]}
  | c r1 r2 ⇒⟦r1⟧ ·⟦r2⟧
  | o r1 r2 ⇒⟦r1⟧ ∪⟦r2⟧
  | k r1 ⇒⟦r1⟧* ].
```

6.3 Pointed regular expressions

A pointed item is a data type used to encode a *set of positions* inside a regular expression. A simple way to formalize pointers inside a data type is by means of a labelled version of the very same data type. For our purposes, it is enough to mark positions preceding individual characters, so we shall have two kinds of characters •*a* (*pp a*) and *a* (*ps a*) according to whether *a* is pointed or not.

```
inductive pitem (S: DeqSet) : Type[0] :=
   pz: pitem S
 | pe: pitem S
 | ps: S → pitem S
 | pp: S → pitem S
 | pc: pitem S → pitem S → pitem S
 | po: pitem S → pitem S → pitem S
 | pk: pitem S → pitem S.
```

A *pointed regular expression* (PRE) is just a pointed item with an additional boolean, that must be understood as the possibility of having a trailing point *at the end* of the expression. As we shall see, pointed regular expressions can be understood as states of a DFA, and the boolean indicates if the state is final or not.

```
definition pre :=λS.pitem S × bool.
```

The *carrier* |*i*| of an item *i* is the regular expression obtained from *i* by removing all the points. Similarly, the *carrier* of a pointed regular expression is the carrier of its item. The formal definition of this functions are straightforward, so we omit them.

The intuitive semantics of a point is as a mark on the position where we should start reading the regular expression. The language associated to a PRE is the union of the languages associated with its points. Here is the straightforward definition (the question mark is an implicit parameter):

```
let rec semi (S : DeqSet) (i : pitem S) on i : word S → Prop :=
match r with
[ pz ⇒ ∅
| pe ⇒ ∅
| ps _ ⇒ ∅
| pp x ⇒ {[x]}
| pc i1 i2 ⇒ ((semi ? i1) ·⟦|i2|⟧) ∪(semi ? i2)
| po i1 r2 ⇒ (semi ? i1) ∪(semi ? i2)
| pk i1 ⇒ (semi ? i1) ·⟦|i1|⟧* ].

definition semp :=λS : DeqSet.λp:pre S.
  if (\snd p) then semi ? (\fst p) ∪{ϵ} else semi ? (\fst p).
```

In the sequel, we shall often use the same notation for functions defined over re, items or PREs, leaving to the reader the simple disambiguation task. Matita is also able to solve automatically this kind of notational overloading. We shall denote with ⟦*e*⟧ all semantic functions *sem*, *semi* and *semp*.

Example 2.

(*1*) *If* e *contains no point then* ⟦e⟧=∅

(*2*) $[\![(a+\bullet bb)^*]\!] = [\![bb(a+bb)^*]\!]$

□

Here are a few, simple, semantic properties of items

```
lemma not_epsilon_item : ∀S:DeqSet.∀i:pitem S.
      ¬ (⟦i⟧ ϵ).
lemma epsilon_pre : ∀S.∀i:pre S.
      (⟦i⟧ ϵ) ↔ (\snd i = true).
lemma minus_eps_item: ∀S.∀i:pitem S.
      ⟦i⟧ ≐ ⟦i⟧-{ϵ}.
lemma minus_eps_pre: ∀S.∀i:pre S.
      ⟦\fst i⟧ ≐ ⟦i⟧-{ϵ}.
```

The first property is proved by a simple induction on i; the other results are easy corollaries.

6.4 Intensional equality of PREs

Items and PREs are a very concrete datatype: they can be effectively compared, and enumerated. This is important, since PREs *are* the states of our finite automata, and we shall need to compare states for bisimulation in Section 6.10.

We can define `beqitem` and `beqitem_true` enriching the set `(pitem S)` to a `DeqSet`.

```
definition DeqItem :=λS.
  mk_DeqSet (pitem S) (beqitem S) (beqitem_true S).
```

Matita's mechanism of *unification hints* [6], see Section 5.6, allows the type inference system to look at `(pitem S)` as the carrier of `DeqSet`, and at `beqitem` as if it were the equality function of `DeqSet`.

The product of two DeqSets is clearly still a DeqSet. Via unification hints, we may enrich a product type to the corresponding DeqSet; since moreover the type of booleans is a DeqSet too, this means that the type of PREs *automatically inherits* the structure of a DeqSet (in Section 6.10, we shall deal with pairs of PREs, and in this case too, without having anything to declare, the type will inherit the structure of a DeqSet).

Items and PREs can also be enumerated. In particular, it is easy to define a function `pre_enum` that takes as input a regular expression and gives back the list of all PREs having `e` for carrier. Completeness of `pre_enum` is stated by the following lemma:

```
lemma pre_enum_complete : ∀S.∀e:pre S.
  memb ? e (pre_enum S (|\fst e|)) = true.
```

6.5 Broadcasting points

Intuitively, a regular expression e must be understood as a pointed expression with a single point in front of it. However, since we only allow points before symbols, we must broadcast this initial point inside e traversing all nullable subexpressions (i.e. subexpressions denoting languages that accept the empty string), which essentially corresponds to the ϵ-closure operation on automata. We use the notation $\bullet(\cdot)$ to denote such an operation; its definition is the expected one: let us start discussing an example.

Example 3. *Let us broadcast a point inside* $(a+\epsilon)(b^*a+b)b$. *We start working in parallel on the first occurrence of* a *(where the point stops), and on* ϵ *that gets* *traversed. We have hence reached the end of* $a+\epsilon$ *and we must pursue broadcasting inside* $(b^* a+b)b$*. Again, we work in parallel on the two additive subterms* $b^* a$ *and* b*; the first point is allowed to both enter the star, and to traverse it, stopping in front of* a*; the second point just stops in front of* b*. No point reached the end of* $b^* a+b$ *hence no further propagation is possible. In conclusion:*

$$\bullet((a+\epsilon)(b^* a+b)b)=\langle(\bullet a+\epsilon)((\bullet b)^* \bullet a+\bullet b)b, false\rangle$$

□

Note that broadcasting a point inside an item generates a PRE, since the point could possibly reach the end of the expression.

Broadcasting inside a pair i_1+i_2 amounts to broadcasting in parallel inside i_1 and i_2. If we define

$$\langle i_1, b_1\rangle \oplus \langle i_2, b_2\rangle = \langle i_1+i_2, b_1 \vee b_2\rangle$$

then, we just have $\bullet(i_1+i_2) = \bullet(i_1) \oplus \bullet(i_2)$.

Concatenation is a bit more complex. In order to broadcast an item inside i_1Âůi_2 we should start broadcasting it inside i_1 and then proceed into i_2 if and only if a point reached the end of i_1.

This suggests to define $\bullet(i_1 \cdot i_2)$ as $\bullet(i_1) \triangleright i_2$, where $e \triangleright i$ is a general operation of concatenation between a PRE and item (named *pre_concat_l*) defined by cases on the boolean in e

$$\langle i_1, true\rangle \triangleright i_2 = i_1 \triangleleft \bullet(i\_2) \qquad \langle i_1, false\rangle \triangleright i_2 = \langle i_1 \cdot i_2, false\rangle$$

In turn, $\triangleleft$ (named *pre_concat_r*) says how to concatenate an item with a PRE, which is extremely simple:

$$i_1 \triangleleft \langle i_1, b\rangle = \langle i_1 \cdot i_2, b\rangle$$

The different kinds of concatenation between items and PREs are summarized in Fig. 3, where we also depict the concatenation between two PREs of Section 6.8.

The definition of $\bullet(\cdot)$ (*eclose*) and $\triangleright$ (*pre_concat_l*) are mutually recursive. In

	item	pre
item	$i_1 \cdot i_2$	$i_1 \triangleleft e_2$ $i_1 \triangleleft \langle i_2, b\rangle := \langle i_1 \cdot i_2, b\rangle$
pre	$e_1 \triangleright i_2$ $\langle i_1, true\rangle \triangleright i_2 := i_1 \triangleleft \bullet(i_2)$ $\langle i_1, false\rangle \triangleright i_2 := \langle i_1 \cdot i_2, false\rangle$	$e_1 \odot e_2$ $e_1 \odot \langle i_2, b\rangle :=$ let $\langle i', b'\rangle := e_1 \triangleright i_2$ in $\langle i', b \vee b'\rangle$

Fig. 3. Concatenations between items and PREs and respective equations

this situation, a viable alternative that is usually simpler to reason about, is to abstract one of the two functions with respect to the other.

```
definition pre_concat_l :=
λS. λbcast:(∀S.pitem S → pre S). λe1:pre S. λi2:pitem S.
  let ⟨i1,b1⟩ := e1 in
  if b1 then i1 ◁(bcast ? i2) else ⟨i1 ·i2,false⟩.
```

```
let rec eclose (S: DeqSet) (i: pitem S) on i : pre S :=
 match i with
  [ pz ⇒ ⟨pz S, false⟩
  | pe ⇒ ⟨pe S, true⟩
  | ps x ⇒ ⟨ps S x, false⟩
  | pp x ⇒ ⟨pp S x, false⟩
  | po i1 i2 ⇒ •i1 ⊕ •i2
  | pc i1 i2 ⇒ •i1 ▷i2
  | pk i ⇒ ⟨(fst (•i))^*,true⟩ ].
```

The definition of *eclose* can then be *lifted* from items to PREs:

```
definition lift :=λS.λf:pitem S →pre S.λe:pre S.
  let ⟨i,b⟩ := e in ⟨\fst (f i), \snd (f i) ∨ b⟩.

definition preclose :=λS. lift S (eclose S).
```

By induction on the item *i* it is easy to prove the following result:

```
lemma erase_bullet : ∀S.∀i:pitem S. |\fst (•i)| = |i|.
```

6.6 Semantics

We are now ready to state the main semantic properties of ⊕ , ▷ , ◁ and •(·):

```
lemma sem_oplus: ∀S:DeqSet.∀e1,e2:pre S.
  ⟦e1 ⊕ e2⟧ ≐ ⟦e1⟧ ∪ ⟦e2⟧.

lemma sem_pre_concat_r : ∀S,i.∀e:pre S.
  ⟦i ◁ e⟧ ≐ ⟦i⟧  · ⟦|fst e|⟧ ∪ ⟦e⟧.

lemma sem_pre_concat_l : ∀S.∀e1:pre S.∀i2:pitem S.
  ⟦e ▷ i⟧ ≐  ⟦e⟧  · ⟦|i|⟧ ∪ ⟦i⟧.

theorem sem_bullet: ∀S:DeqSet. ∀i:pitem S.
  ⟦•i⟧ ≐ ⟦i⟧ ∪ ⟦|i|⟧.
```

The proofs of *sem_oplus*} and *sem_pre_concat_r* are straightforward. For the others, we proceed as follow: we first prove the following auxiliary lemma, that assumes *sem_bullet*

```
lemma sem_pre_concat_l_aux : ∀S.∀e1:pre S.∀i2:pitem S.
   ⟦•i2⟧ ≐  ⟦i2⟧ ∪ ⟦|i2|⟧ →
     ⟦e1 ▷ i2⟧ ≐ ⟦e1⟧  · ⟦|i2|⟧ ∪ ⟦i2⟧.
```

Then, using the previous result, we prove *sem_bullet* by induction on *i*. Finally, *sem_pre_concat_l_aux* and *sem_bullet* give *sem_pre_concat_l*.

It is important to observe that all proofs have an algebraic flavor. Let us consider for instance the proof of *sem_pre_concat_l_aux*. Assuming $e_1 = \langle i_1, b_1\rangle$ we proceed by case-analysis on b_1. If b_1 is false, the result is trivial; if b_1 is true, we have

$$\begin{array}{ll}
[\![\langle i_1, true\rangle \triangleleft i_2]\!] \doteq [\![i_1]\!] \triangleright \bullet(i_2) & \text{by def. of } \triangleleft \\
\doteq [\![i_1]\!] \cdot [\![|fst\ \bullet(i_2)|]\!] \cup [\![\bullet(i_2)]\!] & \text{by } sem\_pre\_concat\_r \\
\doteq [\![i_1]\!] \cdot [\![|i_2|]\!] \cup [\![i_2]\!] \cup [\![|i_2|]\!] & \text{by } erase\_bullet \text{ and } sem\_bullet \\
\doteq [\![i_1]\!] \cdot [\![|i_2|]\!] \cup [\![|i_2|]\!] \cup [\![i_2]\!] & \text{by assoc. and comm.} \\
\doteq ([\![i_1]\!] \cup \{\epsilon\}) \cdot [\![|i_2|]\!] \cup [\![i_2]\!] & \text{by } distr\_cat\_r \\
\doteq [\![\langle i_1, true\rangle]\!] \cdot [\![|i_2|]\!] \cup [\![i_2]\!] & \text{by the semantics of pre}
\end{array}$$

As another example, let us consider the proof of *sem_bullet*}. The proof is by induction on i; let us consider the case of $i_1 \cdot i_2$. We have:

$$\begin{array}{ll}
[\![\bullet(i_1 \cdot i_2)]\!] \doteq [\![\bullet(i_1)]\!] \triangleleft [\![i_2]\!] & \text{by definition of } \bullet(\cdot) \\
\doteq [\![\bullet(i_1)]\!] \cdot [\![|i_2|]\!] \cup [\![i_2]\!] & \text{by } sem\_pre\_concat\_l \\
\doteq ([\![i_1]\!] \cup [\![|i_1|]\!]) \cdot [\![|i_2|]\!] \cup [\![i_2]\!] & \text{by induction hypothesis} \\
\doteq [\![i_1]\!] \cdot [\![|i_2|]\!] \cup [\![|i_1|]\!] \cdot [\![|i_2|]\!] \cup [\![i_2]\!] & \text{by } distr\_cat\_r \\
\doteq ([\![i_1]\!] \cdot [\![|i_2|]\!] \cup [\![i_2]\!]) \cup [\![|i_1 \cdot i_2|]\!] & \text{by assoc. and comm.} \\
\doteq [\![(i_1 \cdot i_2)]\!] \cup [\![|i_1 \cdot i_2|]\!] & \text{by definition of } [\![\_]\!]
\end{array}$$

6.7 Initial state

As a corollary of theorem *sem_bullet*, given a regular expression e, we can easily find an item with the same semantics of e: it is enough to get an item (*blank S e*) having e as carrier and no point, and then broadcast a point in it:

$$[\![\bullet(blank\ S\ e)]\!] \doteq [\![(blank\ S\ e)]\!] \cup [\![e]\!] \doteq [\![e]\!]$$

The definition of *blank* is straightforward; its main properties (both proved by an easy induction on e) are the following:

```
lemma forget_blank: ∀S.∀e:re S.|blank S e| = e.
lemma sem_blank: ∀S.∀e:re S. ⟦blank S e⟧ ≐∅.
theorem re_embedding: ∀S.∀e:re S. ⟦•(blank S e)⟧ ≐⟦e⟧.
```

6.8 Lifted operators

plus and *eclose* have been already lifted from items to PREs. We can now do a similar job for concatenation (⊙) and and Kleene's star ⊛).

```
definition lifted_cat :=λS:DeqSet.λe:pre S.lift S (pre_concat_l S eclose e).

definition lk :=λS:DeqSet.λe:pre S.
  let ⟨i1,b1⟩ := e in if b1 then ⟨(fst (eclose ? i1))*, true⟩ else ⟨i1*,false⟩.
```

We can easily prove the following properties:

```
lemma sem_odot: ∀S.∀e1,e2: pre S.
  ⟦e1 ⊙ e2⟧ ≐ ⟦e1⟧ · ⟦|fst e2|⟧∪⟦e2⟧.

theorem sem_ostar: ∀S.∀e:pre S.
  ⟦e⊛⟧ ≐ ⟦e⟧ · ⟦|fst e|⟧*.
```

For example, let us look at the proof of the latter. Given $e=\langle i,b\rangle$ we proceed by case-analysis on b. If b is false the result is trivial; if b is true we have:

$$\begin{aligned}
\llbracket\langle i,true\rangle^{\circledast}\rrbracket &\doteq \llbracket(fst\ \bullet(i))^*\rrbracket\cup\{\epsilon\} && \text{by definition of } {}^{\circledast}\\
&\doteq \llbracket fst\ \bullet(i)\rrbracket\cdot\llbracket fst\ |\bullet(i)|\rrbracket^*\cup\{\epsilon\} && \text{by definition of } \llbracket\_\rrbracket\\
&\doteq \llbracket fst\ \bullet(i)\rrbracket\cdot\llbracket|i|\rrbracket^*\cup\{\epsilon\} && \text{by } \mathit{erase\_bullet}\\
&\doteq (\llbracket\bullet(i)\rrbracket-\{\epsilon\})\cdot\llbracket|i|\rrbracket^*\cup\{\epsilon\} && \text{by } \mathit{minus\_eps\_pre}\\
&\doteq ((\llbracket i\rrbracket\cup\llbracket|i|\rrbracket)-\{\epsilon\})\cdot\llbracket|i|\rrbracket^*\cup\{\epsilon\} && \text{by } \mathit{sem\_bullet}\\
&\doteq ((\llbracket i\rrbracket-\{\epsilon\})\cup(\llbracket|i|\rrbracket-\{\epsilon\}))\cdot\llbracket|i|\rrbracket^*\cup\{\epsilon\} && \text{by } \mathit{distr\_minus}\\
&\doteq (\llbracket i\rrbracket\cup(\llbracket|i|\rrbracket-\{\epsilon\}))\cdot\llbracket|i|\rrbracket^*\cup\{\epsilon\} && \text{by } \mathit{minus\_eps\_item}\\
&\doteq \llbracket i\rrbracket\cdot\llbracket|i|\rrbracket^*\cup(\llbracket|i|\rrbracket-\{\epsilon\})\cdot\llbracket|i|\rrbracket^*\cup\{\epsilon\} && \text{by } \mathit{distr\_cat\_r}\\
&\doteq \llbracket i\rrbracket\cdot\llbracket|i|\rrbracket^*\cup\llbracket|i|\rrbracket^* && \text{by } \mathit{star\_fix\_eps}\\
&\doteq (\llbracket i\rrbracket\cup\{\epsilon\})\cdot\llbracket|i|\rrbracket^* && \text{by } \mathit{distr\_cat\_r}\\
&\doteq \llbracket\langle i,true\rangle\rrbracket\cdot\llbracket|i|\rrbracket^* && \text{by definition of } \llbracket\_\rrbracket
\end{aligned}$$

6.9 Moves

We now define the move operation, corresponding to the advancement of the state in response to the processing of an input character a. The intuition is clear: we have to look at points preceding the given character a, allow the point to traverse the character, and broadcast it. All other points must be removed.

We can give a particularly elegant definition in terms of the lifted operators of the previous section:

```
let rec move (S: DeqSet) (x:S) (E: pitem S) on E : pre S :=
 match E with
  [ pz ⇒ ⟨pz S, false⟩
  | pe ⇒ ⟨pe S, false⟩
  | ps y ⇒ ⟨ps S y, false⟩
  | pp y ⇒ ⟨ps S, x == y⟩(*the point is advanced if x==y, erased otherwise*)
  | po e1 e2 ⇒ (move ? x e1) ⊕ (move ? x e2)
  | pc e1 e2 ⇒ (move ? x e1) ⊙ (move ? x e2)
  | pk e ⇒ (move ? x e)⊛ ].
```

Example 4. *Let us consider the PRE* $(\bullet a+\epsilon)((\bullet b)^*\bullet a+\bullet b)\ b$ *and the two moves w.r.t. the characters* a *and* b. *For* a, *we have two possible positions (all other points gets erased); the innermost point stops in front of the final* b, *the other one is broadcasted inside* $(b^* a+b)\ b$, *so*

$$move\ a\ ((\bullet a+\epsilon)((\bullet b)^*\bullet a+\bullet b)b) = \langle(a+\epsilon)((\bullet b)^*\bullet a+\bullet b)\bullet b, false\rangle$$

For b, *we have two positions too. The innermost point stops in front of the final* b *too, while the other point reaches the end of* b^* *and must go back through* $b^* a$:

$$move\ b\ ((\bullet a+\epsilon)((\bullet b)^*\bullet a+\bullet b)b) = \langle(a+\epsilon)((\bullet b)^*\bullet a+b)\bullet b, false\rangle$$

□

Obviously, a move does not change the carrier of the item, as one can easily prove by induction on the item

```
lemma same_carrier: ∀S:DeqSet.∀a:S.∀i:pitem S.
  |fst (move a i)| = |i|.
```

Here is our first major result.

```
theorem move_ok: ∀S:DeqSet.∀a:S.∀i:pitem S.
   [[move a i]] ≐ deriv [[i]] a.
```

The proof is a simple induction on i. Let us see the case of concatentation:

$$\begin{array}{lll}
[\![move\ a\ (i_1\cdot i_2)]\!] & \doteq [\![move\ a\ i_1 \odot move\ a\ i_2]\!] & \text{by def. of } move \\
& \doteq [\![move\ a\ i_1]\!]\cdot[\![|fst\ (move\ a\ i_2)|]\!]\cup[\![move\ a\ i_2]\!] & \text{by } sem\_odot \\
& \doteq [\![move\ a\ i_1]\!]\cdot[\![|i_2|]\!]\cup[\![move\ a\ i_2]\!] & \text{by } same\_carrier \\
& \doteq (deriv\ [\![i_1]\!]a)\cdot[\![|i_2|]\!]\cup(deriv\ [\![i_2]\!]a) & \text{by ind. hyp.} \\
& \doteq (deriv\ ([\![i_1]\!]\cdot[\![|i_2|]\!])\ a)\cup(deriv\ [\![i_2]\!]a) & \text{by } deriv\_cat \\
& \doteq deriv\ ([\![i_1]\!]\cdot[\![|i_2|]\!]\cup[\![i_2]\!])\ a & \text{by } deriv\_union \\
& \doteq deriv\ [\![i_1\cdot i_2]\!]a & \text{by definition of } [\![\_]\!]
\end{array}$$

The move operation is generalized to strings in the obvious way:

```
let rec moves (S : DeqSet) w e on w : pre S :=
 match w with
  [ nil ⇒ e
  | cons a tl ⇒ moves S tl (move S a (fst e))].

lemma same_carrier_moves: ∀S:DeqSet.∀w.∀e:pre S.
  |fst (moves ? w e)| = |fst e|.

theorem decidable_sem: ∀S:DeqSet.∀w: word S. ∀e:pre S.
   (snd (moves ? w e) = true) ↔[[e]] w.
```

The proof of *decidable_sem* is by induction on w. The case w=ϵ is trivial; if w=a::w_1 we have

$$\begin{array}{lll}
snd\ (moves\ (a::w_1)\ e) = true & & \\
\quad \leftrightarrow snd\ (moves\ w_1 (move\ a\ (fst\ e))) = true & \text{by def. of } moves \\
\quad \leftrightarrow [\![move\ a\ (fst\ e)]\!]w_1 & \text{by ind. hyp.} \\
\quad \leftrightarrow [\![e]\!]\ a::w_1 & \text{by } move\_ok
\end{array}$$

It is now clear that we can build a deterministic finite automaton (DFA) D_e for e by taking PREs as states, and *move* as transition function; the initial state is $\bullet(e)$ and a state $\langle i,b\rangle$ is final if and only if b=*true*. The fact that states in D_e are finite is obvious: in fact, their cardinality is at most 2^{n+1} where n is the number of symbols in e. This is one of the advantages of pointed regular expressions w.r.t. derivatives, whose finite nature only holds after a suitable quotient.

Example 5. *Figure 4 depicts the DFA for the regular expression* $(ac+bc)^*$. *The graphical description of the automaton is the traditional one, with nodes for states and labelled arcs for transitions. Unreachable states are not shown. Final states are emphasized by a double circle: since a state* $\langle e,b\rangle$ *is final if and only if* b *is true, we may just label nodes with the item.*

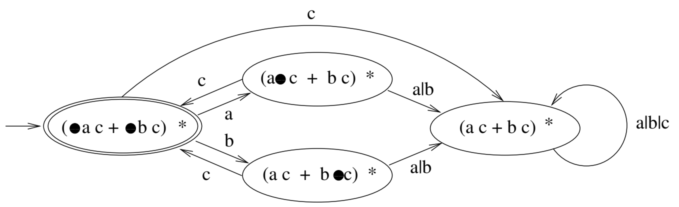


Fig. 4. DFA for `(ac+bc)`*

The automaton is not minimal: it is easy to see that the two states corresponding to the PREs `(a•c +bc)`* *and* `(ac+b•c)`* *are equivalent. A way to prove it is to observe that they define the same language! In fact, each state has a clear semantics given in terms of the associated PRE* `e` *and not of the behaviour of the automaton. As a consequence, the construction of the automaton is not only* direct, *but also extremely* intuitive *and* locally verifiable. □

Example 6. *Starting from the regular expression* `(a+ε)(b`*`a+b)b`, *we obtain the automaton in Figure 5.*

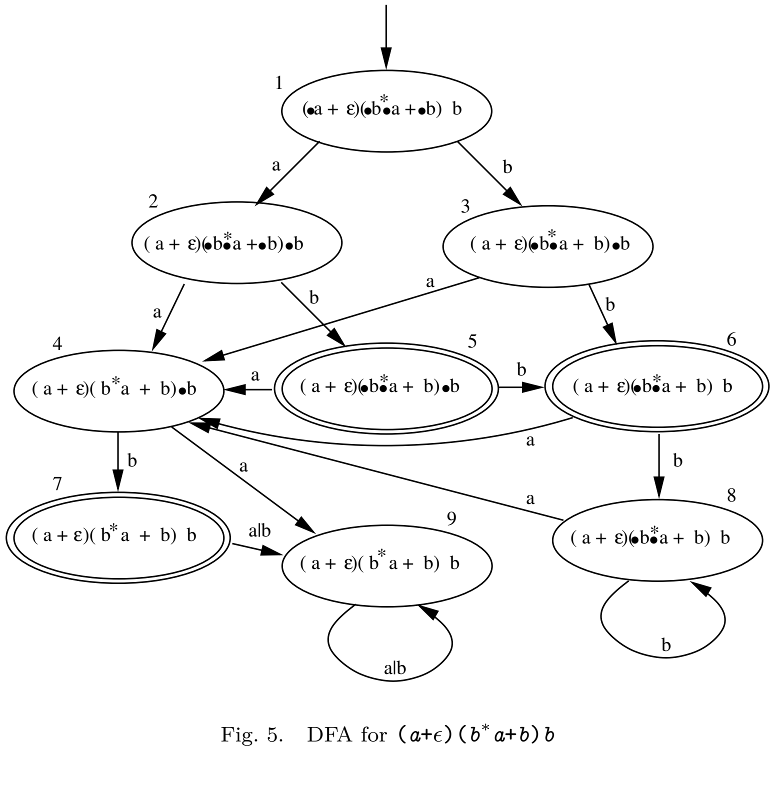


Fig. 5. DFA for `(a+ε)(b`*`a+b)b`

Remarkably, this DFA is minimal, testifying to the small number of states produced by our technique (the pair of states $6-8$ *and* $7-9$ *differ for the fact that* 6 *and* 7 *are final, while* 8 *and* 9 *are not).* □

6.10 Regular expression equivalence

We say that two PREs $\langle i_1, b_1\rangle$ and $\langle i_2, b_2\rangle$ are *cofinal* if and only if $b_1 = b_2$.
As a corollary of `decidable_sem`, we have that two expressions e_1 and e_2 are equivalent iff for any word w the states reachable through w are cofinal.

```
theorem equiv_sem: ∀S:DeqSet.∀e1,e2:pre S.
  ⟦e1⟧ ≐ ⟦e2⟧ ↔ ∀w.cofinal ⟨moves w e1,moves w e2⟩.
```

This does not directly imply decidability: we have no bound on the length of w; moreover, so far, we made no assumption on the cardinality of S. Instead of requiring S to be finite, we may restrict the analysis to characters occurring in the given PREs. This means we can prove the following, stronger result:

```
lemma equiv_sem_occ: ∀S.∀e1,e2:pre S.(∀w.(sublist S w (occ S e1 e2))→
  cofinal ⟨moves w e1,moves w e2⟩) → ⟦e1⟧ ≐⟦e2⟧.
```

The proof essentially requires the notion of sink state and a few trivial properties:

```
definition sink_pre :=λS.λi.⟨blank S (|i|), false⟩.

lemma not_occur_to_sink: ∀S,a.∀i:pitem S. memb S a (occur S (|i|)) ≠true →
  move a i = sink_pre S i.

lemma moves_sink: ∀S,w,i. moves w (sink_pre S i) = sink_pre S i.
```

Let us say that a list of pairs of PREs is a *bisimulation* if it is closed w.r.t. moves, and all its members are cofinal.

```
definition sons :=λS:DeqSet.λl:list S.λp:(pre S)×(pre S).
  map ?? (λa.⟨move a (fst (fst p)),move a (fst (snd p))⟩) l.

definition is_bisim :=λS:DeqSet.λl:list ?.λalpha:list S. ∀p:(pre S)×(pre S).
  memb ? p l = true → cofinal ? p ∧ (sublist ? (sons ? alpha p) l).
```

Using lemma `equiv_sem_occ` it is easy to prove

```
lemma bisim_to_sem: ∀S:DeqSet.∀l:list ?.∀e1,e2: pre S.
  is_bisim S l (occ S e1 e2) → memb ? ⟨e1,e2⟩ l = true → ⟦e1⟧ ≐⟦e2⟧.
```

This is already an interesting result: checking if l is a bisimulation is decidable, hence we could generate l with some untrusted piece of code and then run a (boolean version of) `is_bisim` to check that it is actually a bisimulation. However, in order to prove that equivalence of regular expressions is *decidable* we must prove that we can always effectively build such a list (or find a counterexample). The idea is that the list we are interested in is just the set of all pair of PREs *reachable* from the initial pair via some sequence of moves.

The algorithm for computing reachable nodes in a graph is a very traditional one. We split nodes in two disjoint lists: a list of *visited* nodes and a *frontier*, composed of all nodes connected to a node in visited but not visited already. At each step we select a node a from the frontier, compute its childrens, add a to the set of visited nodes and the (not already visited) childrens to the frontier.

Instead of first computing reachable nodes and then performing the bisimilarity test we can directly integrate it into the algorithm: the set of visited nodes is closed by construction w.r.t. reachability, so we just have to check cofinality for any node we add to visited.

Here is the extremely simple algorithm

```
let rec bisim S l n (frontier,visited: list ?) on n :=
  match n with
  [ O ⇒ ⟨false,visited⟩ (* unreachable code *)
  | S m ⇒
    match frontier with
    [ nil  ⇒ ⟨true,visited⟩
    | cons hd tl  ⇒
      if beqb (snd (fst hd)) (snd (snd hd)) (* cofinality *) then
        bisim S l m
         (unique_append ?
           (filter ? (λx.notb (memb ? x (hd::visited))) (sons S l hd))
           tl)
         (hd::visited)
      else ⟨false,visited⟩
    ]
  ].
```

The integer *n* is an upper bound to the number of recursive calls, equal to the dimension of the graph. It returns a pair composed of a boolean and the set of visited nodes; the boolean is true if and only if all visited nodes are cofinal.

The main test function is:

```
definition equiv :=λSig.λre1,re2:re Sig.
  let e1 :=•(blank ? re1) in
  let e2 :=•(blank ? re2) in
  let n :=S (length ? (space_enum Sig (|fst e1|) (|fst e2|))) in
  let sig :=(occ Sig e1 e2) in
  (bisim ? sig n [⟨e1,e2⟩] []).
```

We proved both correctness and completeness; so, we have

```
theorem equiv_sem : ∀Sig.∀e1,e2:re Sig.
   fst (equiv ? e1 e2) = true ↔⟦e1⟧ ≐⟦e2⟧.
```

For correctness, we use the invariant that at each call of *bisim* the two lists *visited* and *frontier* only contain nodes reachable from $\langle e_1, e_2\rangle$: hence it is absurd to suppose the existence of a pair that is not cofinal. For completeness, we use the invariant that all the nodes in visited are cofinal, and the childrens of *visited* are either in *visited* or in the *frontier*; since at the end *frontier* is empty, *visited* is therefore a bisimulation. All in all, correctness and completeness take little more than a few hundred lines.

7. QUOTIENTING IN TYPE THEORY

One fundamental operation in set theory is quotienting: given a set S and an equivalence relation R over S, quotienting creates a new set S/R whose elements are equivalence classes of elements of S. The idea behind quotienting is to replace the structural equality over S with R, therefore identifying elements up to R. The use of equivalence classes is just a technicality.

Matita does not have the type-theoretic counterpart to quotienting, which are quotient types. In the literature there are alternative proposals to introduce quotient types, but no consensus has been reached. Nevertheless, the notion of setoids delivers all the features of quotient types without needing to extend the type theory. A *setoid* is defined as a type S coupled with an equivalence relation R, used to compare elements of S. In place of working with equivalence classes of S up to R, one keeps working with elements of S, but compares them using R in place of $=$. Setoids over types (elements of `Type[0]`) can be declared in Matita as follows.

```
record equivalence_relation (A: Type[0]) : Type[0] :={
    eqrel:> relation A
  ; equiv_refl: reflexive ... eqrel
  ; equiv_sym: symmetric ... eqrel
  ; equiv_trans: transitive ... eqrel
}.

record setoid : Type[1] :={
    carrier:> Type[0]
  ; eq_setoid: equivalence_relation carrier
}.
```

Note that carrier has been defined as a coercion so that when `S` is a setoid we can write `x:S` in place of `x: carrier S`.

We use the notation ≃ for the equality on setoid elements.

```
notation "hvbox(n break ≃ m)"
  non associative with precedence 45
for @{ 'congruent $n $m }.

interpretation "eq_setoid" 'congruent n m = (eqrel ? (eq_setoid ?) n m).
```

Example: integers are traditionally defined as pairs (n, m) of natural numbers quotiented by $(n1, m1) \simeq (n2, m2)$ iff $n1 + m2 = m1 + n2$. The integer $+n$ is represented by any pair $(k, n + k)$, while the integer $-n$ by any pair $(n + k, n)$.

If we write

```
definition Z: setoid :=
 mk_setoid (ℕ ×ℕ)
  (mk_equivalence_relation ...
   (λc1,c2. \fst c1 + \snd c2 = \fst c2 + \snd c1) ...).
```

we are left with three proof obligations (i.e. three open goals), corresponding to reflexivity, symmetry and transitivity of the given relation. We can immediately close two of them by automation, while transitivity requires some algebraic manipulation:

```
 whd // * #x1 #x2 * #y1 #y2 * #z1 #z3 #H1 #H2
 cut (x1 + y2 + y1 + z3 = y1 + x2 + z1 + y2) // #H3
 cut ((y2 + y1) + (x1 + z3) = (y2 + y1) + (z1 + x2)) // #H4
 @(injective_plus_r ...H4)
qed.
```

The two integers $(0, 1)$ and $(1, 2)$ are equal up to $\simeq$, written ⟨0,1⟩ $\simeq$ ⟨1,2⟩. Unfolding the notation, that corresponds to

eqrel ? (eq_setoid ?) ⟨0,1⟩⟨1,2⟩

which means that the two pairs are to be compared according to the equivalence relation of an unknown setoid ? whose carrier is $\mathbb{N} \times \mathbb{N}$. A hint can be used to always pick Z as “intended” setoid for $\mathbb{N} \times \mathbb{N}$.

```
unification hint 0 ≔ ;
    X ≟ Z
(* ---------------------------------------- *) ⊢
    ℕ × ℕ ≡ carrier X.
```

For instance, thanks to the hint, Matita accepts the following statement:

```
example ex1: ⟨0,1⟩ ≃ ⟨1,2⟩.
```

Every type is a setoid when elements are compared up to Leibniz equality.

```
definition LEIBNIZ: Type[0] → setoid :=
 λA.
  mk_setoid A
   (mk_equivalence_relation ...(eq ...) ...).
 //
qed.
```

As before, a hint can be used to enrich a type to a “LEIBNIZ” setoid:

```
unification hint 10 ≔ A: Type[0];
    X ≟ LEIBNIZ A
(* ---------------------------------------- *) ⊢
    A ≡ carrier X.
```

Note that we declare the previous hint with a lower precedence level (10 vs 0, precedence levels are in decreasing order). In this way an ad-hoc setoid hint will be always preferred to the Leibniz one. for example, ⟨0,1⟩ $\simeq$ ⟨1,2⟩ is still interpreted in Z, while 1 $\simeq$ 2 is interpreted as 1=2.

As another example, propositions up to equivalence form a setoid:

```
definition PROP: setoid :=
 mk_setoid Prop
  (mk_equivalence_relation ...(λx,y. x ↔y) ...).
 whd [ @iff_trans | @iff_sym | /2/ ]
qed.

unification hint 0 := ;
    X ≟ PROP
(* -------------------------------------- *) ⊢
    Prop ≡ carrier X.
```

In set theory functions and relations over a quotient S/R can be defined by lifting a function/relation over S that respects R. Respecting R means that the relation holds for an element x of S iff it holds for every y of S such that xRy. Similarly, a function f respects R iff $f\ x = f\ y$ for every x, y such that xRy.

We say that a function between two setoids is proper when it respects their equalities.

```
definition proper: ∀I,O:setoid. (I → O) → Prop :=
 λI,O,f. ∀x,y:I. x ≃ y → f x ≃ f y.
```

A proper function is called a morphism.

```
record morphism (I,O: setoid) : Type[0] :={
   mor_carr:1> I → O
 ; mor_proper: proper ...mor_carr
 }.
```

We introduce a notation for morphisms using the symbol of an arrow followed by a dot.

```
notation "hvbox(I break →· O)"
  right associative with precedence 20
for @{ 'morphism $I $O }.

interpretation "morphism" 'morphism I O = (morphism I O).
```

By declaring `mor_carr` as a coercion using `:1>`, it is possible to write `(f x)` for `(mor_carr f x)` in order to apply a morphism `f` to an argument. The 1 in `:1>` is the number of arguments required to trigger the coercion.

As a simple example, let us define the negation `opp` of an integer number. We first implement the function over `Z` and then lift it to a morphism. The proof obligation is to prove properness.

```
definition opp: Z → Z :=λc.⟨\snd c,\fst c⟩.

definition OPP: Z →· Z :=mk_morphism ... opp ....
 normalize //
qed.
```

When writing expressions over *Z* we will always use the function *opp*, which does not carry additional information. The following hints will be automatically used by the system to retrieve the morphism associated to *opp* when needed, i.e. to retrieve the proof of properness of *opp*. This is a use of unification hints to implement automatic proof search. The first hint is used when the function is partially applied, the second one when it is applied to an argument.

```
unification hint 0 ≔ ;
    X ≟ OPP
(* ---------------------------------------- *) ⊢
    opp ≡ mor_carr ... X.

unification hint 0 ≔ x:Z ;
    X ≟ OPP
(* ---------------------------------------- *) ⊢
    opp x ≡ mor_carr ... X x.
```

For instance, consider the proof that *opp* is proper. We can prove it without any explicit mention of *OPP*. In particular, when we apply the universal property *mor_proper* of morphisms, Matita looks for the morphism associated to *opp x* and finds it thanks to the second unification hint above, completing the proof.

```
example ex2: ∀x,y. x ≃ y → opp x ≃ opp y.
 #x #y #EQ @mor_proper @EQ
qed.
```

The previous definition of proper only deals with unary functions. In type theory n-ary functions are better handled in curried form as unary functions whose output is a function space. When we restrict to morphisms, we do not need a notion of properness for n-ary functions because the function space can also be seen as a setoid quotienting functions using functional extensionality: two morphisms are equal when they map equal elements to equal elements.

```
definition function_space: setoid → setoid → setoid :=
 λI,O.
  mk_setoid (I →· O)
   (mk_equivalence_relation ... (λf,g. ∀x,y:I. x ≃ y → f x ≃ g y) ...).
 whd
 [ #f1 #f2 #f3 #f1_f2 #f2_f3 #x #y #EQ @(equiv_trans ... (f2 x)) /2/
 | #f1 #f2 #f1_f2 #x #y #EQ @(equiv_sym ... (f1_f2 ...)) @equiv_sym //
 | #f #x #y #EQ @mor_proper // ]
qed.

unification hint 0 ≔ I,O: setoid;
    X ≟ function_space I O
(* ---------------------------------------- *) ⊢
    I →· O ≡ carrier X.
```

We overload the notation so that *I →· O* can mean both the type of morphisms from *I* to *O* or the function space from *I* to *O*.

```
interpretation "function_space" 'morphism I O = (function_space I O).
```

A binary morphism is just obtained by currying. In the following we will use *I1 →· I2 →· O* directly in place of *bin_morphism*.

```
definition bin_morphism: setoid → setoid → setoid → Type[0] :=
 λI1,I2,O. I1 →· I2 →· O.
```

Directly constructing an inhabitant of a binary morphism is annoying because one needs to write a function that returns a morphism instead of a binary function. Moreover, there are two proof obligations to prove. We can simplify the work by introducing a new constructor for binary morphisms that takes in input a binary function and opens a single proof obligation, called proper2.

```
definition proper2: ∀I1,I2,O: setoid. (I1 → I2 → O) → Prop :=
 λI1,I2,O,f.
  ∀x1,x2,y1,y2. x1 ≃ x2 → y1 ≃ y2 → f x1 y1 ≃ f x2 y2.

definition mk_bin_morphism:
 ∀I1,I2,O: setoid. ∀f: I1 → I2 → O. proper2 … f → I1 →· I2 →· O :=
λI1,I2,O,f,proper.
  mk_morphism … (λx. mk_morphism … (λy. f x y) …) ….
 normalize /2/
qed.
```

We can also coerce a binary morphism to a binary function and prove that *proper2* holds for every binary morphism.

```
definition binary_function_of_binary_morphism:
 ∀I1,I2,O: setoid. (I1 →· I2 →· O) → (I1 → I2 → O) :=
λI1,I2,O,f,x,y. f x y.

coercion binary_function_of_binary_morphism:
 ∀I1,I2,O: setoid. ∀f:I1 →· I2 →· O. (I1 → I2 → O) :=
 binary_function_of_binary_morphism
 on _f: ? →· ? →· ? to ? → ? → ?.

theorem mor_proper2: ∀I1,I2,O: setoid. ∀f: I1 →· I2 →· O. proper2 … f.
 #I2 #I2 #O #f #x1 #x2 #y1 #y2 #EQx #EQy @(mor_proper … f … EQx … EQy)
qed.
```

Continuing our example on integer numbers, let us define addition. As usual, we first define it as a function and then lift it to a morphism (we overload + over integers):

```
definition Zplus: Z → Z → Z :=λx,y. ⟨\fst x + \fst y,\snd x + \snd y⟩.

interpretation "Zplus" 'plus x y = (Zplus x y).

definition ZPLUS: Z →· Z →· Z :=mk_bin_morphism … Zplus ….
 normalize * #x1a #x1b * //
qed.
```

The following hint allow the system to automatically retrieve the morphism ZPLUS associated with Zplus, as needed.

```
unification hint 0 ≔ x,y:Z ;
    X ≟ ZPLUS
(* --------------------------------------- *) ⊢
    x + y ≡ mor_carr ...X x y.
```

The identity function is a morphism and composition of morphisms is also a morphism. This means that the identity function is always proper and a compound context is proper if every constituent is.

```
definition id_morphism: ∀S: setoid. S →· S :=
 λS. mk_morphism ...(λx.x) .... //
qed.

definition comp_morphism: ∀S1,S2,S3. (S2 →· S3) → (S1 →· S2) → (S1 →· S3) :=
 λS1,S2,S3,f1,f2. mk_morphism ...(λx. f1 (f2 x)) ....
 normalize #x1 #x2 #EQ @mor_proper @mor_proper //
qed.
```

By iterating applications of *mor_proper*, we can consume the context one application at a time in order to perform a rewriting. Like in *ex2*, the proof works on any setoid because it does not reference OPP anywhere. The above theorem on composition of morphisms justifies the correctness of the proofs.

```
example ex3: ∀x,y. x ≃ y → opp (opp (opp x)) ≃ opp (opp (opp y)).
 #x #y #EQ @mor_proper @mor_proper @mor_proper @EQ
qed.
```

We can improve the readability of the previous proof by assigning a notation to *mor_proper* and by packing together the various applications of *mor_proper* and *EQ*. We pick the prefix symbol †.

```
notation "† c" with precedence 90 for @{'proper $c }.
interpretation "mor_proper" 'proper c = (mor_proper ????? c).

example ex3': ∀x,y. x ≃ y → opp (opp (opp x)) ≃ opp (opp (opp y)).
 #x #y #EQ @(†(†(†EQ)))
qed.
```

While the term `(†(†(†EQ)))` can seem puzzling at first, note that it closely matches the term `(opp (opp (opp x)))`. Each occurrence of the unary morphism *opp* is replaced by † and the occurrence *x* to be rewritten to *y* is replaced by *EQ*: $x \simeq y$. Therefore the term `(†(†(†EQ)))` is a compact notation to express at once where the rewriting should be performed and what hypothesis should be used for rewriting. In the next section we address the problem of setoid rewriting in full generality.

7.1 Rewriting setoid equalities

In set theory, once a quotient S/R has been defined, equivalence classes are compared using the standard equality $=$, whose main property is to allow rewriting in every context: if $E_1 = E_2$ then $f\ E_1$ can be replaced with $f\ E_2$ in every context f. Note that f is applied to equivalence classes of elements of S. If we started instead from a function f' defined on elements of S, first of all we would need to lift it to work on equivalence classes, and this is possible only if f' is a *proper context* for R, defined as an f' such that $s_1 R s_2$ implies $f'\ s_1 = f'\ s_2$.

When using setoids, we keep working with elements of `S` instead of forming a new type. Therefore, we must deal with contexts `f` that are not proper. When `f` is not proper, `f E1` cannot be replaced with `f E2` even if `E1` $\simeq$ `E2`. For example, ⟨`0,1`⟩ $\simeq$ ⟨`1,2`⟩ but `\fst` ⟨`0,1`⟩ $\neq$ `\fst` ⟨`1,2`⟩. Therefore every time we want to rewrite `E1` with `E2` under the assumption that `E1` $\simeq$ `E2` we need to *prove the context to be proper*. Most of the time the context is just a composition of morphisms and, like in `ex3'`, the only information that the user needs to give to the system is the position of the occurrences of `E1` to be replaced and the equations to be used for the rewriting. As for `ex3'`, we can provide a simple syntax to describe contexts and equations at the same time. The syntax is just given by a few notations to hide applications of `mor_proper`, reflexivity, symmetry and transitivity.

Here is a synopsis of the syntax:

†_	to rewrite in the argument of a unary morphism
_ ‡ _	to rewrite in both arguments of a binary morphism
#	to avoid rewriting in this position
t	to rewrite from left to right in this position using the proof *t*. Usually t is the name of an hypothesis in the context of type `E1` $\simeq$ `E2`
`t^-1`	to rewrite from right to left in this position using the proof t. Concretely, it applies symmetry to *t*, proving `E2` $\simeq$ `E1` from `E1` $\simeq$ `E2`.
._	to start rewriting when the goal `P E1` is not an equation $\simeq$, and the goal to be obtained is `P E2`. Concretely, it applies the proof of `P E2` $\rightarrow$ `P E1` obtained by splitting the double implication `P E2` $\leftrightarrow$ `P E1`, which is equivalent to `P E2` $\simeq$ `P E1` where $\simeq$ is the equality of the `PROP` setoid. Thus the argument should be a proof of `P E2` $\simeq$ `P E1`, obtained using the previous symbols according to the shape of `P`.
.=_	to prove an equation `G1` $\simeq$ `G2` by first rewriting into `E1` leaving a new goal `G1'` $\simeq$ `G2`. Concretely, it applies transitivity of $\simeq$.

```
notation "l ‡ r" with precedence 90 for @{'proper2 $l $r }.
interpretation "mor_proper2" 'proper2 x y =
   (mor_proper ? (function_space ? ?) ?? ? x ?? y).

notation "#" with precedence 90 for @{'reflex}.
interpretation "reflexivity" 'reflex = (equiv_refl ???).

interpretation "symmetry" 'invert r = (equiv_sym ???? r).

notation ".= r" with precedence 55 for @{'trans $r}.
interpretation "transitivity" 'trans r = (equiv_trans ????? r ?).
```

```
notation ". r" with precedence 55 for @{'fi $r}.
definition fi: ∀A,B:Prop. A ≃ B → (B → A) :=λA,B,r. proj2 ?? r.
interpretation "fi" 'fi r = (fi ?? r).
```

The following example shows several of the features at once:

(1) the first occurrence of *x2* is turned into *x1* by rewriting the hypothesis from right to left.
(2) the first occurrence of *x1* is turned into *x2* by rewriting the hypothesis from left to right.
(3) the two rewritings are performed at once.
(4) the subterm *z+y* does not need to be rewritten. Therefore a single `#` is used in place of `#‡#`, which is also correct but produces a larger proof.
(5) we can directly start with an application of ‡ because the goal is a setoid equality

```
example ex4: ∀x1,x2,y,z:Z. x1 ≃ x2 →
 (y + x2) + (x1 + (z + y)) ≃ (y + x1) + (x2 + (z + y)).
 #x1 #x2 #y #z #EQ @((#‡EQ^-1)‡(EQ‡#))
qed.
```

The following example is just to illustrate the use of `.=` We prove the same statement of *ex4*, but this time we perform one rewriting at a time. Note that in an intermediate goal Matita replaces occurrences of *Zplus* with occurrences of (the carrier of) *ZPLUS*. To recover the notation `+` it is sufficient to expand the declaration of *ZPLUS*.

```
example ex5: ∀x1,x2,y,z:Z. x1 ≃ x2 →
 (y + x2) + (x1 + (z + y)) ≃ (y + x1) + (x2 + (z + y)).
 #x1 #x2 #y #z #EQ @(.=(#‡EQ^-1)‡#) whd in match ZPLUS; @(#‡(EQ‡#))
qed.
```

Our last example involves rewriting under a predicate different from ≃ . We first introduce such a predicate over integers.

```
definition is_zero: Z → Prop :=λc. \fst c = \snd c.

definition IS_ZERO: Z →· PROP :=mk_morphism ... is_zero ....
 normalize /3 by conj,injective_plus_r/
qed.

unification hint 0 ≔ x:Z ;
    X =?= IS_ZERO
(* ---------------------------------------- *) ⊢
    is_zero x ≡ mor_carr ... X x.
```

We can rewrite under any predicate using "`.`".

```
example ex6: ∀x,y:Z. x ≃ y → is_zero (x + y) → is_zero (x + x).
 #x #y #EQ #H @(.†(#‡EQ)) @H
qed.
```

7.2 Dependent setoids

A setoid is essentially a type equipped with its own notion of equality. In a framework with dependent types, one expects to be able to build setoids dependent on other types and setoids. Working with families of setoids that depend on a plain type — i.e. not another setoid — poses no additional problem. For example, we can build a setoid out of vectors of length *n* assigning to it the type $\mathbb{N} \to$ *setoid*. All the machinery for setoids just introduced keeps working. On the other hand, types that depend over a setoid require much more complex machinery and, in practice, it is not advised to try to work with them in an intentional type theory like the one of Matita.

To understand the issue, imagine that we have defined a family of types I dependent over integers: I: $Z \to Type[0]$. Because $\langle 0,1\rangle$ and $\langle 1,2\rangle$ both represent the same integer +1, the two types $I\ \langle 0,1\rangle$ and $I\ \langle 1,2\rangle$ should have exactly the same inhabitants. However, being different types, their inhabitants are disjoint. The solution is to equip the type I with a transport function t: $\forall n,m{:}Z.\ n \simeq m \to I\ n \to I\ m$ that maps an element of $I\ n$ to the corresponding element of $I\ m$. Starting from this idea, the picture quickly becomes complex when one start considering all the additional equations that the transport functions should satisfy. For example, if p: $n \simeq m$, then `t ...p (t ...p^-1 x) = x`, i.e. the element in $I\ n$ corresponding to the element in $I\ m$ that corresponds to x in $I\ n$ should be exactly x. Moreover, for any function f: $\forall n.\ I\ n \to T\ n$ for some other type T dependent on n the following equation should hold: `f ...(t ...p x) = t ...p (f ...x)` (i.e. transporting and applying f should commute because f should be insensitive too up to $\simeq$ to the actual representation of the integral indexes).

Luckily enough, types dependent on setoids are rare in practice. Most examples of dependent types are indexed over discrete objects, like natural, integers and rationals, that admit a unique representation. By continuity, types dependent on real numbers can also be represented as types dependent on a dense subset of the reals, like the rational numbers.

7.3 Avoiding setoids

Quotients are used pervasively in mathematics. In many practical situations, for example when dealing with finite objects like pairs of naturals, a unique representation can be imposed, for example by introducing a normalization procedure to be called after every operation. For example, integer numbers can be normalized to either $\langle 0,n\rangle$ or $\langle n,0\rangle$. Or they can be represented as either 0 or a non zero number, with the latter being encoded by a boolean (the sign) and a natural (the predecessor of the absolute value). For example, -3 would be represented by *NonZero* $\langle$*false*,2$\rangle$, +3 by *NonZero* $\langle$*true*,2$\rangle$ and 0 by *Zero*. Rational numbers n/m can be put in normal form by dividing both n and m by their greatest common divisor, or by picking n=0, m=1 when n is 0.

Imposing a unique representation is not always possible. For example, picking a canonical representative for a Cauchy sequence is not a computable operation. Nevertheless, when possible, avoiding setoids is preferable:

(1) when Leibniz equality is used, replacing n with m knowing n=m does not require any proof of properness;

(2) at the moment automation in Matita is only available for Leibniz equalities. By switching to setoids fewer proofs are automatically found;

(3) types dependent on plain types do not need ad-hoc transport functions because the rewriting principle for Leibniz equality plays that role and already satisfies for free all required equations;

(4) normal forms are usually smaller than other forms. By sticking to normal forms both the memory consumption and the computational cost of operations may be reduced.

8. INFINITE STRUCTURES AND COINDUCTIVE TYPES

The only primitive data types of Matita are dependent products and universes. So far, every other user defined data type has been an inductive type. An inductive type is declared by giving its list of constructors (or introduction rules in the case of predicates). An inhabitant of an inductive type is obtained by composing together a finite number of constructors and data of other types, according to the type of the constructors. Therefore all inhabitants of inductive types are essentially finite objects (obviously, up to the possible arguments of constructors). Natural numbers, lists, trees, states a DFA, letters of an alphabet are all finite and can be defined inductively.

Sometimes, however, it may be necessary to represent and manipulate infinite structures. Typical examples are: sequences, real numbers (a special case of sequences), data streams (e.g. as read from a network interface), traces of diverging computations of a program, etc.

In the next section we shall start discussing a few examples exploiting functions for the representation of objects; then, we shall move on to a different encoding, based on coinductive types.

8.1 Real Numbers as Cauchy sequences

In many cases it is possible to describe an infinite object by means of an infinite sequence of approximations (also called observations), where the sequence can be simply defined as a function on the domain of natural numbers.

```
definition seq : Type[0] → Type[0] :=λA:Type[0]. ℕ→ A.
```

A well known example is the definition of Real numbers by means of Cauchy sequences. First of all, we axiomatize the relevant properties of rational numbers.

```
axiom Q: Type[0].
axiom Qdist: Q → Q → Q.
axiom symmetric_Qdist: ∀x,y. Qdist x y = Qdist y x.
axiom Qleq: Q → Q → Prop.
interpretation "Qleq" 'leq x y = (Qleq x y).
axiom transitive_Qleq: transitive ... Qleq.
axiom Qplus: Q → Q → Q.
interpretation "Qplus" 'plus x y = (Qplus x y).
axiom Qleq_Qplus: ∀qa1,qb1,qa2,qb2.
 qa1 ≤ qb1 → qa2 ≤ qb2 → qa1 + qa2 ≤ qb1 + qb2.
axiom Qdist_Qplus: ∀qa1,qb1,qa2,qb2.
 Qdist (qa1 + qa2) (qb1 + qb2) ≤ Qdist qa1 qb1 + Qdist qa2 qb2.
axiom Qhalve: Q → Q.
axiom Qplus_Qhalve_Qhalve: ∀q. Qhalve q + Qhalve q = q.
axiom triangular_Qdist: ∀x,y,z. Qdist x z ≤ Qdist x y + Qdist y z.
```

Then, we can define a sequence of rationals

```
definition Qseq: Type[0] :=seq Q.
```

and state the Cauchy property.

```
definition Cauchy: Qseq → Prop :=
 λR:Qseq. ∀eps. ∃n. ∀i,j. n ≤ i → n ≤ j → Qdist (R i) (R j) ≤ eps.
```

A real number is an equivalence class of Cauchy sequences. In type theory we can define *R* as a setoid whose carrier *R_* is the type of Cauchy sequences.

```
record R_: Type[0] :=
 { r: Qseq
 ; isCauchy: Cauchy r
 }.
```

Two sequences are equal when for every epsilon they are eventually pointwise closer than epsilon.

```
definition R : setoid :=
 mk_setoid R_
  (mk_equivalence_relation ...
   (λr1,r2:R_. ∀eps. ∃n. ∀i. n ≤ i → Qdist (r r1 i) (r r2 i) ≤ eps) ...).
[ (* transitivity *) #x #y #z #H1 #H2 #eps cases (H1 (Qhalve eps)) #i -H1 #H1
  cases (H2 (Qhalve eps)) #j -H2 #H2 %{(max i j)} #l #Hmax
  <(Qplus_Qhalve_Qhalve eps) @transitive_Qleq
  [3: @(Qleq_Qplus ...(H1 l ...) (H2 l ...)) /2 by le_maxl,le_maxr/
  |2: // ]
| (* symmetry *) #x #y #H #eps cases (H eps) /3 by ex_intro/
| (* reflexivity *) #x #eps cases (isCauchy x eps) /3 by ex_intro/ ]
qed.

unification hint 0 ≔ ;
    X ≟ R
(* ---------------------------------------- *) ⊢
    R_ ≡ carrier R.
```

The following coercion is used to write *r n* to extract the n-th approximation from the real number *r*.

```
coercion R_to_fun : ∀r:R. ℕ→ Q :=r on _r:R to ?.
```

We conclude this first example considering the definition of the sum operation between real numbers. This just requires pointwise addition of the approximations. The proof that the resulting sequence is Cauchy is the standard one: to obtain an approximation up to ϵ it is necessary to approximate both summands up to $\epsilon/2$.

```
definition Rplus_: R_ → R_ → R_ :=
 λr1,r2. mk_R_ (λn. r1 n + r2 n) ....
 #eps
 cases (isCauchy r1 (Qhalve eps)) #n1 #H1
 cases (isCauchy r2 (Qhalve eps)) #n2 #H2
 %{(max n1 n2)} #i #j #K1 #K2 @(transitive_Qleq ...(Qdist_Qplus ...))
 <(Qplus_Qhalve_Qhalve eps) @Qleq_Qplus [@H1 @le_maxl | @H2 @le_maxr]
 [2,6: @K1 |4,8: @K2]
qed.
```

We lift *Rplus_* to a morphism by proving that it respects equivalence of real numbers.

We closely follow the usual mathematical proof: to compute the sum up to epsilon it is sufficient to fetch the arguments at precision $\epsilon/2$.

```
definition Rplus: R →· R →· R :=
 mk_bin_morphism ...Rplus_ ....
 #x1 #x2 #y1 #y2 #Hx #Hy #eps
 cases (Hx (Qhalve eps)) #i -Hx #Hx
 cases (Hy (Qhalve eps)) #j -Hy #Hy
 %{(max i j)} #l #Hmax <(Qplus_Qhalve_Qhalve eps) @transitive_Qleq
 [3: @(Qleq_Qplus ...(Hx l ...) (Hy l ...)) /2 by le_maxl,le_maxr/
 |2: // ]
qed.
```

8.2 Traces of a program

Let's introduce a very simple programming language whose instructions can test and set a single implicit variable.

```
inductive instr: Type[0] :=
   p_set: ℕ→ instr              (* sets the variable to a constant *)
 | p_incr: instr                (* increments the variable *)
 | p_while: list instr → instr. (* repeats until the variable is 0 *)
```

The states of a program are given by the current value of the variable and the list of instructions to be executed.

```
definition state :=ℕ×(list instr).
```

The following function defines the transition relation beteen states:

```
inductive next: state → state → Prop :=
   n_set: ∀n,k,o. next ⟨o,(p_set n)::k⟩ ⟨n,k⟩
 | n_incr: ∀k,o. next ⟨o,p_incr::k⟩ ⟨S o,k⟩
 | n_while_exit: ∀b,k. next ⟨0,(p_while b)::k⟩ ⟨0,k⟩
 | n_while_loop: ∀b,k,o. next ⟨S o,(p_while b)::k⟩ ⟨S o,b@(p_while b)::k⟩.
```

A diverging trace is a sequence of states such that the n+1-th state is obtained executing one step from the n-th state.

```
record div_trace: Type[0] :=
 { div_tr: seq state
 ; div_well_formed: ∀n. next (div_tr n) (div_tr (S n))
 }.
```

The previous definition of trace is not computable: we cannot write a program that given an initial state returns its trace.

To write such a function, we first write a computable version of next, called step.

```
definition step: state → option state :=
 λs. let ⟨o,k⟩ :=s in
  match k with
   [ nil  ⇒ None ?
   | cons hd k  ⇒
      Some ...match hd with
      [ p_set n  ⇒ ⟨n,k⟩
      | p_incr  ⇒ ⟨S o,k⟩
      | p_while b  ⇒
         match o with
         [ O  ⇒ ⟨0,k⟩
         | S p  ⇒ ⟨S p,b@(p_while b)::k⟩ ]]].
```

We can esasiliy prove that `next o n` if and only if `step o = Some ...n`:

```
theorem step_next: ∀o,n. step o = Some ...n → next o n.
theorem next_step: ∀o,n. next o n → step o = Some ...n.
```

Termination is the archetypal undecidable problem. Therefore there is no function that given an initial state returns a diverging trace if the program diverges or fails in case of error. The best we can do is to give an alternative definition of trace that captures both diverging and converging computations.

```
record trace: Type[0] :=
 { tr: seq (option state)
 ; well_formed: ∀n,s. tr n = Some ...s → tr (S n) = step s
 }.
```

The trace is diverging if every state is not final.

```
definition diverging: trace → Prop :=λt. ∀n. tr t n ≠None ?.
```

We leave as a simple exercise for the reader to check that the two definitions of diverging traces are equivalent, as expressed by the following statements:

```
theorem div_trace_to_diverging_trace:
 ∀d: div_trace. ∃t: trace. diverging t ∧ tr t 0 = Some ...(div_tr d 0).

theorem diverging_trace_to_div_trace:
 ∀t: trace. diverging t → ∃d: div_trace. tr t 0 = Some ...(div_tr d 0).
```

While we cannot decide divergence, we can always compute a trace from a given initial state. Note that every time the n-th element of the trace is accessed, all the elements in position ≤ n are computed too.

```
let rec compute_trace_seq (s:state) (n:nat) on n : option state :=
 match n with
 [ O ⇒ Some ...s
 | S m ⇒
    match compute_trace_seq s m with
    [ None  ⇒ None ...
    | Some o  ⇒ step o ]].
```

We can emphasize that the initial state of the trace computed by the previous function is s by returning a suitable sigma type:

```
definition compute_trace: ∀s:state. Σt:trace. tr t 0 = Some ...s.
 #s %
 [ %{(compute_trace_seq s)}
   #n #o elim n [ whd in ⊢(??%? →??%?); #EQ destruct // ]
   -n #n #_ #H whd in ; whd in ⊢(??%?); >H //
 | // ]
qed.
```

8.3 Infinite data types as coinductive types

All the previous examples were handled very easily via sequences declared as functions. A few criticisms can be made to this approach though:

(1) the sequence allows for random access. In many situations, however, the elements of the sequence are meant to be read one after the other, in increasing order of their position. Moreover, the elements are meant to be consumed (read) linearly, i.e. just once. This suggests a more efficient implementation where sequences are implemented with state machines that emit the next value and enter a new state every time they are read. Indeed, some programming languages like OCaml differentiate between (possibly infinite) lists, that are immutable, from mutable streams whose only access operation yields the head and turns the stream into its tail. Data streams read from the network are a typical example of streams: the previously read values are not automatically memoized and are lost if not saved when read. Files on disk are also usually read via streams to avoid keeping all of them in main memory. Another typical case where streams are used is that of diverging computations: instead of generating at once an infinite sequence of outputs, a function is turned into a stream and asking the next element of the stream runs one more iteration of the function to produce the next output (often an approximation).

(2) if a sequence computes the n-th entry by recursively calling itself on every entry less than n, accessing the n-th entry requires re-computation of all entries in position $\leq$ n, which is very inefficient.

(3) by representing an infinite object as a collection of approximations, the structure of the object is lost. This was not evident in the previous examples because in all cases the intrinsic structure of the datatype was just that of being ordered and sequences capture the order well. Imagine, however, that we want to represent an infinite binary tree of elements of type A with the previous technique. We need to associate to each position in the infinite tree an element of type A. A position in the tree is itself a path from the root to the node of interest. Therefore the infinite tree is represented as the function $(\mathbb{N} \rightarrow \mathbb{B}) \rightarrow A$ where $\mathbb{B}$ are the booleans and the tree structure is already less clear. Suppose now that the binary tree may not be full, i.e. some nodes can have fewer than two children. Representing such a tree is definitely harder. We may for example use the data type $(\mathbb{N} \rightarrow \mathbb{B}) \rightarrow option\ A$ where $None$ would be associated to the position $b_1 \ldots b_n$ if a node in such position does not exist. However, we would

need to add the invariant that if $b_1 \ldots b_n$ is undefined (i.e. assigned to *None*), so are all suffixes $b_1 \ldots b_n\ b_{n+1} \ldots b_{n+j}$.

The previous observations suggest the need for primitive infinite datatypes in the language, usually called coinductive types or codata. Matita allows us to declare coinductive types with the same syntax used for inductive types, just replacing the keyword **inductive** with **coinductive**. Semantically, the two declarations differ because a coinductive type also contains infinite inhabitants (or, more precisely, finite, cyclic generators for them) that respect the typechecking rules.

As a simple example, instead of using functions, an infinite sequence of elements of type *A* can be defined by the following coinductive definition:

```
coinductive streamseq (A: Type[0]) : Type[0] :=
 sscons: A → streamseq A → streamseq A.
```

Note that we do not have a "base case", so how can we inhabit the previous type? In general, coinductive types are inhabited by infinite data obtained by means of a special kind of coinductive definitions, introduced by the keyword **let corec**.

The syntax of **let corec** definitions is the same of **let rec** definitions, but for the declarations of the recursive argument. While **let rec** definitions are recursive definition that are strictly decreasing on the recursive argument, **let corec** definitions are productive recursive definitions. A recursive definition is productive if, when its definiendum is fully applied to its arguments, it reduces in a finite amount of time to (the application of) a constructor of a coinductive type. In typical cases, the definiendum is already the application of a constructor.

Let us see some simple examples of coinductive definitions of corecursive streamseqs. The following definition defines an infinite sequence of zeros.

```
let corec all_zeros: streamseq nat :=sscons nat 0 all_zeros.
```

Note that *all_zeros* is not a function and does not have any argument. The definition is clearly productive because its definiendum is the constructor *sscons*.

The following definition defines the sequence $n, n+1, n+2, \ldots$ starting from an input value n.

```
let corec from_n (n:ℕ) : streamseq nat :=sscons ...n (from_n (S n)).
```

The definition essentially behaves like an automaton with state *n*. When the streamseq is pattern matched, the current value *n* is returned as head of the streamseq and the tail of the streamseq is the automaton with state (*S n*).

In order to retrieve the n-th element from a streamseq we can write a function recursive on *n*.

```
let rec streamseq_nth (A: Type[0]) (s: streamseq A) (n:ℕ) on n : A :=
 match s with [ sscons hd tl ⇒
  match n with [ O ⇒ hd | S m ⇒ streamseq_nth ...tl m ]].
```

Any sequence can be turned into the equivalent streamseq and the other way around.

```
let corec streamseq_of_seq (A: Type[0]) (s: seq A) (n:ℕ) : streamseq A :=
 sscons ...(s n) (streamseq_of_seq A s (S n)).

lemma streamseq_of_seq_ok:
 ∀A:Type[0]. ∀s: seq A. ∀m,n.
  streamseq_nth A (streamseq_of_seq ...s n) m = s (m+n).
 #A #s #m elim m normalize //
qed.

definition seq_of_streamseq: ∀A:Type[0]. streamseq A → seq A :=streamseq_nth.

lemma seq_of_streamseq_ok:
 ∀A:Type[0]. ∀s: streamseq A. ∀n. seq_of_streamseq ...s n = streamseq_nth ...
s n.
 //
qed.
```

8.4 Real numbers via coinductive types

In thi section, we closely follow the content of Section 8.1, replacing sequences with streamseqs.

```
definition Qstreamseq: Type[0] :=streamseq Q.
definition Qstreamseq_nth :=streamseq_nth Q.
```

Here is the Cauchy property

```
definition Cauchy': Qstreamseq → Prop :=
 λR:Qstreamseq. ∀eps. ∃n. ∀i,j. n ≤ i → n ≤ j →
  Qdist (Qstreamseq_nth R i) (Qstreamseq_nth R j) ≤ eps.
```

and the definition of a real numbers as an equivalence class of Cauchy sequences:

```
record R_': Type[0] :=
 { r': Qstreamseq
 ; isCauchy': Cauchy' r'
 }.
```

The following statement provides the setoid structure for reals; the proof that *Cauchy'* is an equivalence relation is identical to the proof is section 8.1, and we skip it.

```
definition R' : setoid :=
 mk_setoid R_'
  (mk_equivalence_relation ...
   (λr1,r2:R_'. ∀eps. ∃n. ∀i. n ≤ i →
    Qdist (Qstreamseq_nth (r' r1) i) (Qstreamseq_nth (r' r2) i) ≤ eps) ...).

unification hint 0 ≔ ;
    X ≟ R'
(* ---------------------------------------- *) ⊢
    R_' ≡ carrier R'.
```

The following coercion is used to write *r n* to extract the n-th approximation from the real number *r*.

```
coercion R_to_fun' : ∀r:R'. ℕ→ Q :=(λr. Qstreamseq_nth (r' r)) on _r:R' to ?.
```

Pointwise addition over *Qstreamseq* defined by corecursion. The definition is productive because, when *Rplus_streamseq* is applied to two closed values of type *Qstreamseq*, it will reduce to *sscons*.

```
let corec Rplus_streamseq (x:Qstreamseq) (y:Qstreamseq) : Qstreamseq :=
 match x with [ sscons hdx tlx ⇒
 match y with [ sscons hdy tly ⇒
  sscons ...(hdx + hdy) (Rplus_streamseq tlx tly) ]].
```

The following lemma was for free using sequences. In the case of streamseqs it must be proved by induction on the index because *Qstreamseq_nth* is defined by recursion on the index.

```
lemma Qstreamseq_nth_Rplus_streamseq:
 ∀i,x,y.
   Qstreamseq_nth (Rplus_streamseq x y) i
  = Qstreamseq_nth x i + Qstreamseq_nth y i.
 #i elim i [2: #j #IH] * #xhd #xtl * #yhd #ytl // normalize @IH
qed.
```

The proof that the resulting sequence is Cauchy is exactly the same we used for sequences, up to two applications of the previous lemma.

```
definition Rplus_': R_' →R_' →R_' :=
 λr1,r2. mk_R_' (Rplus_streamseq (r' r1) (r' r2)) ....
 #eps
 cases (isCauchy' r1 (Qhalve eps)) #n1 #H1
 cases (isCauchy' r2 (Qhalve eps)) #n2 #H2
 %{(max n1 n2)} #i #j #K1 #K2
 >Qstreamseq_nth_Rplus_streamseq >Qstreamseq_nth_Rplus_streamseq
 @(transitive_Qleq ...(Qdist_Qplus ...))
 <(Qplus_Qhalve_Qhalve eps) @Qleq_Qplus [@H1 @le_maxl | @H2 @le_maxr]
 [2,6: @K1 |4,8: @K2]
qed.
```

We lift *Rplus_'* to a morphism by proving that it respects the equality for real numbers. The script is exactly the same we used to lift *Rplus_*, but for two applications of *Qstreamseq_nth_Rplus_streamseq*.

```
definition Rplus': R' →· R' →· R' :=
 mk_bin_morphism ...Rplus_' ....
 #x1 #x2 #y1 #y2 #Hx #Hy #eps
 cases (Hx (Qhalve eps)) #i -Hx #Hx
 cases (Hy (Qhalve eps)) #j -Hy #Hy
 %{(max i j)} #l #Hmax <(Qplus_Qhalve_Qhalve eps) @transitive_Qleq
 [3: @(Qleq_Qplus ...(Hx l ...) (Hy l ...)) /2 by le_maxl,le_maxr/
 |2: >Qstreamseq_nth_Rplus_streamseq >Qstreamseq_nth_Rplus_streamseq // ]
qed.
```

8.5 Intermezzo: the dynamics of coinductive data

Corecursive definitions, like recursive ones, are a form of definition where the definiens occurs in the definiendum. Matita compares terms up to reduction and reduction always allows the expansion of non recursive definitions. If it also allowed the expansion of recursive definitions, reduction could diverge and type checking would become undecidable. For example, consider again the definition of *all_zeros*

```
let corec all_zeros: streamseq nat :=sscons nat 0 all_zeros.
```

If the system expanded all occurrences of *all_zeros*, it would expand it forever to `sscons ...0 (sscons ...0 (sscons ...0 ...))`.

In order to avoid divergence, recursive definitions are only expanded when a certain condition is satisfied. In the case of a **let rec** defined function *f* that is recursive on its n-th argument, *f* is only expanded when it occurs in an application `(f t1 ... tn ...)` and tn is (the application of) a constructor. Termination is guaranteed by the combination of this restriction and *f* being guarded by destructors: the application `(f t1 ... tn ...)` can reduce to a term that contains another application `(f t1' ... tn' ...)` but the size of *tn'* (roughly the number of nested constructors) will be smaller than the size of tn eventually leading to termination.

Dual restrictions are put on **let corec** definitions. If *f* is a **let rec** defined term, *f* is only expanded when it occurs in the scrutinee of a match expression, that is in a situation of the kind

match *f t1* ... *tn* **with** ...

To better understand the duality, that is not syntactically perfect, note that: in the recursive case *f* destructs terms and is expanded only when applied to a constructor; in the co-recursive case *f* constructs terms and is expanded only when it becomes argument of the match destructor.

Termination is guaranteed by the combination of this restriction and *f* being productive: the term **match** *f t1* ... *tn* ...**with** will reduce in a finite amount of time to a match applied to a constructor, whose reduction can expose another application of *f*, but not another **match** *f t1' .. tn'* ...**with**. Therefore, since no new matches around *f* can be created by reduction, the number of destructors that surrounds the application of *f* decreases at every reduction of the **match**, eventually leading to termination.

Even if a coinductively defined *f* does not reduce in every context to its definiendum, it is possible to prove that the definiens is Leibniz equal to its definiendum. The trick is to prove first an eta-expansion lemma for the coinductive type that states that an inhabitant of the coinductive type is equal to the one obtained destructing and rebuilding it via a match. The proof is done by cases on the inhabitant. Let's see an example.

```
lemma eta_streamseq:
 ∀A:Type[0]. ∀s: streamseq A.
  match s with [ sscons hd tl ⇒ sscons ...hd tl ] = s.
 #A * //
qed.
```

In order to prove now that *all_zeros* is equal to its definiendum, it suffices to rewrite it using the *eta_streamseq* lemma in order to insert around the definiens the match destructor that triggers its expansion.

```
lemma all_zeros_expansion: all_zeros = sscons ...0 all_zeros.
 <(eta_streamseq ? all_zeros) in ⊢(??%?); //
qed.
```

Expansions lemmas as the one just presented are almost always required to progress in non trivial proofs, as we will see in the next example. In contrast, the equivalent expansions lemmas for **let rec** definitions are rarely required.

8.6 Traces of a program via coinductive types

A diverging trace is a streamseq of states such that the n+1-th state is obtained by executing one step from the n-th state. The definition of *div_well_formed'* is the same we already used for sequences, but on streamseqs.

```
definition div_well_formed' : streamseq state → Prop :=
 λs: streamseq state.
  ∀n. next (streamseq_nth ...s n) (streamseq_nth ...s (S n)).

record div_trace': Type[0] :=
 { div_tr':> streamseq state
 ; div_well_formed'': div_well_formed' div_tr'
 }.
```

The well formedness predicate on streamseq can also be redefined using as a coinductive predicate. A streamseq of states is well formed if the second element is the next of the first and the stream without the first element is recursively well formed.

```
coinductive div_well_formed_co: streamseq state → Prop :=
 is_next:
  ∀hd1:state. ∀hd2:state. ∀tl:streamseq state.
   next hd1 hd2 → div_well_formed_co (sscons ...hd2 tl) →
    div_well_formed_co (sscons ...hd1 (sscons ...hd2 tl)).
```

Note that Matita automatically proves the inversion principles for every coinductive type, but no general coinduction principle. That means that the **elim** tactic does not work when applied to a coinductive type. The tactics **inversion** and **cases** are the only ways to work with coinductive hypothesis.

A proof of *div_well_formed* cannot be built stacking a finite number of constructors. The type can only be inhabited by a coinductive definition. As an example, we show the equivalence between the two definitions of well formedness for streamseqs.

A *div_well_formed'* stream is also *div_well_formed_co*. We write the proof term explicitly, omitting the subterms that prove *div_well_formed' (sscond ...hd2 tl)* and *next hd1 hd2*. Therefore we will obtain two proof obligations. The given proof term is productive: if we apply it to a closed term of type streamseq state and a proof that it is well formed, the two matches in head position will reduce and the lambda-abstraction will be consumed, exposing the *is_next* constructor.

```
let corec div_well_formed_to_div_well_formed_co
 (s: streamseq state): div_well_formed' s → div_well_formed_co s :=
 match s with [ sscons hd1 tl1 ⇒
  match tl1 with [ sscons hd2 tl ⇒
   λH: div_well_formed' (sscons ...hd1 (sscons ...hd2 tl)).
    is_next ...
     (div_well_formed_to_div_well_formed_co (sscons ...hd2 tl) ...) ]].
[ (* First proof obligation: next hd1 hd2 *)
  @(H 0)
| (* Second proof obligation: div_well_formed' (sscons ...hd2 tl) *)
  @(λn. H (S n)) ]
qed.
```

A *div_well_formed_co* stream is also *div_well_formed'*. This time the proof is by induction on the index and inversion on the *div_well_formed_co* hypothesis.

```
theorem div_well_formed_co_to_div_well_formed:
 ∀s: streamseq state. div_well_formed_co s → div_well_formed' s.
 #s #H #n lapply H -H lapply s -s elim n [2: #m #IH]
 * #hd1 * #hd2 #tl normalize #H inversion H #hd1' #hd2' #tl' #Hnext #Hwf
 #eq destruct /2/
qed.
```

Like for sequences, because of undecidability of termination there is no function that given an initial state returns the diverging trace if the program diverges or fails in case of error. We need a new data type to represent a possibly finite stream of elements. Such a data type is usually called stream and can be defined elegantly as a coinductive type.

```
coinductive stream (A: Type[0]) : Type[0] :=
   snil: stream A
 | scons: A → stream A → stream A.
```

The following lemma will be used to unblock blocked reductions, as previously explained for *eta_streamseq*.

```
lemma eta_stream:
 ∀A:Type[0]. ∀s: stream A.
  match s with [ snil ⇒ snil ...| scons hd tl ⇒ scons ...hd tl ] = s.
 #A * //
qed.
```

In order to give the formal definition of a trace, we first need to (coinductively) define the *well_formedness* predicate, whose definition is slightly more complex than the previous one.

```
coinductive well_formed': stream state → Prop :=
   wf_snil: ∀s. step s = None ...→ well_formed' (scons ...s (snil ...))
 | wf_scons:
    ∀hd1,hd2,tl.
     step hd1 = Some ...hd2 →
    well_formed' (scons ...hd2 tl) →
     well_formed' (scons ...hd1 (scons ...hd2 tl)).
```

Note: we could have equivalently defined *well_formed'* avoiding coinduction by defining a recursive function to retrieve the n-th element of the stream, if any. From now on we will stick to coinductive predicates only to show more examples of usage of coinduction. In a formalization, however, it is always better to explore several alternatives and see which ones work best for the problem at hand.

```
record trace': Type[0] :=
 { tr':> stream state
 ; well_formed'': well_formed' tr'
 }.
```

The trace is diverging if every state is not final. Again, this can be defined by means of a coinductive definition.

```
coinductive diverging': stream state → Prop :=
 mk_diverging': ∀hd,tl. diverging' tl → diverging' (scons ...hd tl).
```

The two coinductive definitions of diverging traces are equivalent. To state the two results we first need a function to retrieve the head from traces and diverging traces.

```
definition head_of_streamseq: ∀A:Type[0]. streamseq A → A :=
 λA,s. match s with [ sscons hd _ ⇒ hd ].

definition head_of_stream: ∀A:Type[0]. stream A → option A :=
 λA,s. match s with [ snil ⇒ None ...| scons hd _ ⇒ Some ...hd ].
```

A streamseq can be extracted from a diverging stream using corecursion.

```
let corec mk_diverging_trace_to_div_trace'
 (s: stream state) : diverging' s → streamseq state :=
 match s return λs. diverging' s → streamseq state
 with
 [ snil ⇒ λabs: diverging' (snil ...). ?
 | scons hd tl ⇒ λH. sscons ? hd (mk_diverging_trace_to_div_trace' tl ...) ].
 [ cases (?:False) inversion abs #hd #tl #_ #abs' destruct
 | inversion H #hd' #tl' #K #eq destruct @K ]
qed.
```

An expansion lemma will be needed soon.

```
lemma mk_diverging_trace_to_div_trace_expansion:
 ∀hd,tl,H. ∃K.
  mk_diverging_trace_to_div_trace' (scons state hd tl) H =
   sscons ...hd (mk_diverging_trace_to_div_trace' tl K).
 #hd #tl #H cases (eta_streamseq ...(mk_diverging_trace_to_div_trace' ??)) /2/
qed.
```

To complete the proof we need a final lemma: streamseqs extracted from well formed diverging streams are well formed too. The definition is mostly interactive because we need to use the expansion lemma above to rewrite the type of the scons branch. Otherwise, Matita rejects the term as ill-typed.

```
let corec div_well_formed_co_mk_diverging_trace_to_div_trace
 (t : stream state)
:∀W:well_formed' t.
  ∀H:diverging' t.
   div_well_formed_co (mk_diverging_trace_to_div_trace' t H)
:=match t return λt. well_formed' t → diverging' t → ?
 with
 [ snil ⇒ λ_.λabs. ?
 | scons hd tl ⇒ λW.λH. ? ].
[ cases (?:False) inversion abs #hd #tl #_ #eq destruct
| cases (mk_diverging_trace_to_div_trace_expansion …H) #H' #eq
  lapply (sym_eq ??? … eq) #Req cases Req -Req -eq -H
  cases tl in W H';
  [ #_ #abs cases (?:False) inversion abs #hd #tl #_ #eq destruct
  | -tl #hd2 #tl #W #H
    cases (mk_diverging_trace_to_div_trace_expansion …H) #K #eq >eq
    inversion W [ #s #_ #abs destruct ]
    #hd1' #hd2' #tl' #eq1 #wf #eq2 lapply eq1
    cut (hd=hd1' ∧ hd2 = hd2' ∧ tl=tl') [ destruct /3/ ]
    ** -eq2 ***
    #eq1 %
    [ @step_next //
    | <eq @div_well_formed_co_mk_diverging_trace_to_div_trace // ]]]
qed.

theorem diverging_trace_to_div_trace':
 ∀t: trace'. diverging' t →∃d: div_trace'.
  head_of_stream … t = Some …(head_of_streamseq … d).
 #t #H %
 [ %{(mk_diverging_trace_to_div_trace' …H)}
   @div_well_formed_co_to_div_well_formed
   @div_well_formed_co_mk_diverging_trace_to_div_trace
   @(well_formed'' t)
 | cases t in H; * normalize // #abs cases (?:False) inversion abs
   [ #s #_ #eq destruct | #hd1 #hd2 #tl #_ #_ #eq destruct ]]
qed.
```

A stream can be extracted from a streamseq using corecursion.

```
let corec stream_of_streamseq (A: Type[0]) (s: streamseq A) : stream A :=
 match s with [ sscons hd tl ⇒ scons …hd (stream_of_streamseq …tl) ].
```

We need again an expansion lemma to typecheck the proof that the resulting stream is divergent.

```
lemma stream_of_streamseq_expansion:
 ∀A,hd,tl.
    stream_of_streamseq A (sscons …hd tl)
  = scons …hd (stream_of_streamseq …tl).
 #A #hd #tl <(eta_stream ...(stream_of_streamseq ...)) //
qed.
```

The proof that the resulting stream is diverging also needs corecursion.

```
let corec diverging_stream_of_streamseq (s: streamseq state) :
 diverging' (stream_of_streamseq ...s) :=
 match s return λs. diverging' (stream_of_streamseq ...s)
 with [ sscons hd tl ⇒ ? ].
 >stream_of_streamseq_expansion % //
qed.

let corec well_formed_stream_of_streamseq (d: streamseq state) :
 div_well_formed' ...d → well_formed' (stream_of_streamseq state d) :=
 match d
 return λd. div_well_formed' d → ?
 with [ sscons hd1 tl ⇒
  match tl return λtl. div_well_formed' (sscons ...tl) → ?
  with [ sscons hd2 tl2 ⇒ λH.? ]].
 >stream_of_streamseq_expansion >stream_of_streamseq_expansion %2
 [2: <stream_of_streamseq_expansion @well_formed_stream_of_streamseq
   #n @(H (S n))
 | @next_step @(H O) ]
qed.

theorem div_trace_to_diverging_trace':
 ∀d: div_trace'. ∃t: trace'. diverging' t ∧
  head_of_stream ...t = Some ...(head_of_streamseq ...d).
 #d %{(mk_trace' (stream_of_streamseq ...d) ...)}
 [ /2/ | % // cases d * // ]
qed.
```

8.7 How to compare coinductive types

Inhabitants of inductive types are compared using Leibniz's equality that, on closed terms, coincides with convertibility. The situation is radically different in the case of coinductive types. Because of the restrictions for the evaluation strategy (see Section 8.5), two inhabitants of coinductive types, even if closed, may not be convertible (having distinct normal forms), but can be shown to be Leibniz equal.

For example, the following two definitions in normal form produce the same streamseq `0,1,0,1,...` but are not equal because the normal forms are syntactically different.

```
let corec zero_one_streamseq1: streamseq ℕ:=
 sscons ...0 (sscons ...1 zero_one_streamseq1).

let corec one_zero_streamseq: streamseq ℕ:=
 sscons ...1 (sscons ...0 one_zero_streamseq).

definition zero_one_streamseq2: streamseq ℕ:=sscons ...0 one_zero_streamseq.
```

In place of Leibniz equality, the right notion to compare coinductive terms is bisimulation. Two terms *t1*, *t2* are bisimilar if one simulates the other and the other way around, where *t2* simulates *t1* if every observation on *t1* can be performed on *t2* as well, and the observed subterms are co-recursively bisimilar. In practice two coinductive terms are bisimilar if they are the same constructor applied to bisimilar coinductive subterms.

We define bisimilarity for streamseqs using a coinductive predicate.

```
coinductive ss_bisimilar (A: Type[0]) : streamseq A → streamseq A → Prop :=
 sscons_bisim:
  ∀x,tl1,tl2.
   ss_bisimilar ... tl1 tl2 →
    ss_bisimilar ...(sscons ...x tl1) (sscons ...x tl2).
```

The two streams above can be proved to be bisimilar. By using the *eta_streamseq* trick twice, we expand both definitions once so to obtain a proof obligation that asks to show that the stream *sscons* ... 0 (*sscons* ... 1 *zero_one_streamseq1*) is bisimilar to *sscons* ... 0 (*sscons* ... 1 *zero_one_streamseq2*). Then we apply twice the *sscons_bisim* constructor to consume the leading 0 and 1, finding again the original goal. Finally we conclude with the coinductive hypothesis. The proof is productive because, after the two rewriting steps, it reduces to two applications of the *sscons_bisim* constructor, immediately followed by the recursive call.

```
let corec zero_one_streamseq_eq:
 ss_bisimilar ... zero_one_streamseq1 zero_one_streamseq2 :=?.
 <(eta_streamseq ... zero_one_streamseq1) normalize
 <(eta_streamseq ... one_zero_streamseq) normalize
 % % @zero_one_streamseq_eq
qed.
```

We can finally turn streamseqs into a setoid and lift every operation on streamseqs to morphisms. We first prove reflexivity, symmetry and transitivity of bisimulation because each proof must be given using `let corec`.

```
let corec reflexive_ss_bisimilar (A: Type[0]): reflexive ...(ss_bisimilar A)
:=?.
 * #hd #tl % @reflexive_ss_bisimilar
qed.

let corec symmetric_ss_bisimilar (A: Type[0]): symmetric ...(ss_bisimilar A)
:=?.
 * #hd1 #tl1 * #hd2 #tl2 * #hd #tl1' #tl2' #H % @symmetric_ss_bisimilar @H
qed.
```

In the proof of transitivity we face a technical problem. After inverting one of the bisimilarity hypothesis we are left with an equation of the form *sscons* ... *hd1 tl1* = *sscons* ... *hd2 tl2* and we need to infer the two equations *hd1* = *hd2* and *tl1* = *tl2* to conclude the proof. The `destruct` tactic does exactly this, but it produces a proof term that is not recognized to be productive by Matita. Moreover, for technical reasons, checking for productivity is only performed at `qed` time: every step of the proof would be accepted by Matita, but the whole proof would be rejected at `qed`.

The trick to preserve productivity is to cut the two equalities and to use destruct to prove them. In this way the `destruct` tactic is used only in a side proof inside which there is no recursive call, and the whole proof term is recognized to be productive.

```
let corec transitive_ss_bisimilar (A: Type[0]):transitive ...(ss_bisimilar A)
:=?.
 * #hd1 #tl1 * #hd2 #tl2 * #hd3 #tl3
 * #hd #tl1' #tl2' #H1
 #H inversion H #hd' #tl1'' #tl2'' -H #H #EQ1 #_
 cut (hd=hd' ∧ tl1''=tl2') [ destruct /2/ ]
 ** #EQ % @transitive_ss_bisimilar //
qed.
```

```
definition STREAMSEQ: Type[0] → setoid :=
 λA.
  mk_setoid (streamseq A)
   (mk_equivalence_relation ...(ss_bisimilar A) ...).
 //
qed.

unification hint 0 := A:Type[0];
    X ≟ STREAMSEQ A
(* ---------------------------------------- *) ⊢
    streamseq A ≡ carrier X.
```

As an example, we lift *streamseq_nth* to a morphism.

```
definition STREAMSEQ_NTH:
 ∀A: Type[0]. STREAMSEQ A →· (LEIBNIZ ℕ) →· (LEIBNIZ A) :=
 λA. mk_bin_morphism ??? (streamseq_nth A) ....
 #x1 #x2 #n1 #n2 #bisim *
 lapply bisim -bisim lapply x2 -x2 lapply x1 -x1 -n2 elim n1
 [ #x1 #x2 #H inversion H //
 | #m #IH #x1 #x2 #H inversion H #hd #tl1 #tl2 #bisim #EQ1 #EQ2
   destruct @IH //]
qed.
```

Working with setoids introduces technical complications that one would rather avoid. The alternative encoding of streams via sequences does not solve the issue. Sequences are functions and, to capture the mathematical meaning of equality for sequences, functions need to be compare using functional extensionality. Two functions are extensionally equal when they are pointiwse equal, i.e. they map equal inputs to the same outputs. Intensional equality of functions, in contrast, just compares the two codes.

For example, the next two enumerations `0,1,2,...` are extensionally, but not intensionally equal.

```
definition enum_N1: seq ℕ:=λi.i.
definition enum_N2: seq ℕ:=λi.i+0.
```

Functional extensionality is defined in the standard library of Matita as follows.

```
definition exteqF: ∀A,B:Type[0].∀f,g:A→ B.Prop :=λA,B.λf,g.
 ∀a.f a = g a.
```

The proof that the two sequences above are extensionally equal is trivial.

```
lemma enum_N_equal: exteqF ... enum_N1 enum_N2.
 normalize //
qed.
```

Like for bisimulation, to work systematically with functional extensionality we need to turn the type of sequences into a setoid. The two alternatives are actually equivalent with respect to this issue.

8.8 Generic construction principles

When an inductive type is defined Matita automatically proves a generic elimination principle using a recursive definition. Later the user can apply the generic eliminator in the middle of a proof, without the need to make a lemma and prove it using a **let rec**.

For example, the principle generated for a list is the following:

```
 ∀A:Type[0]. ∀P: A → Prop.
  P (nil ...) → (∀hd:A. ∀tl: list A. P tl → P (hd::tl)) →
   ∀l:list A. P l
```

Here the predicate *P* is dependent on the list we are analyzing.
The useless corresponding non dependent principle is:

```
 ∀A:Type[0]. ∀P: Prop.
  P → (∀hd:A. ∀tl: list A. P → P) →
   ∀l:list A. P l
```

that we can rewrite up to currying as

```
 ∀A:Type[0]. ∀P: Prop.
  ((unit + A × list A × P) → P) →
   list A → P
```

and, turning the recursor into an iterator, to the simpler form

```
 ∀A:Type[0]. ∀P: Prop.
  ((unit + A × P) → P) →
   list A → P
```

Thinking of a list as the least fixpoint of the equation

```
   list A = unit + A × list A
```

the above principle says that we can prove *P* as soon as we can prove *P* for every possible shape of the list, replacing *sublists (list A)* with the result *P* of recursive calls of the iterator.

Streams are the greatest fixpoint of the same equation

```
  stream A = unit + A ×stream A
```

Dualizing the idea of a generic destructor for list, we obtain the type of the generic constructor for streams:

```
 ∀A:Type[0]. ∀P: Prop.
  (P → (unit + A ×P)) →
   P → stream A
```

The above principle says that we can build a stream from a proof of *P* as soon as we can observe from *P* what shape the stream has and what *P* will be used to generate the reamining of the stream. The principle is indeed easily provable in Matita.

```
let corec stream_coind (A: Type[0]) (P: Prop) (H: P → Sum unit (A ×P))
 (p:P) : stream A :=
 match H p with
 [ inl _ ⇒ snil A
 | inr cpl ⇒ let ⟨hd,p'⟩ :=cpl in scons A hd (stream_coind A P H p') ].
```

At the moment Matita does not automatically prove any generic construction principle. One reason is that the type of the generated principles is less informative than that for elimination principles. The problem is that in the type theory of Matita it is not possible to generalize the above type to a dependent version, to obtain the exact dual of the elimination principle. In case of the elimination principle, the type *P* depends on the value of the input, and the principle proves the dependent product ∀*l*: *list A. P A*. In the case of the introduction principle, the output is *P* → *list A*. To obtain a dependent version, we would need a dual of the dependent product such that the type of the input is dependent on the value of the output: a notion that is not even meaningful when the function is not injective.

9. LOGICAL RESTRICTIONS

In this section we shall discuss some of the logical restrictions of the logical system required for preserving consistency. At the same time, we shall provide a more detailed and precise exposition of the Calculus of Inductive Constructions. In order to fully understand all nuances of the type system, the reader is invited to read the description of the Matita kernel in [5].

9.1 Positivity in inductive definitions

Not every inductive definition is accepted by Matita. The following is a typical example:

```
inductive D : Type[0] :=
  lam: (D → D) → D.
```

which is rejected by the system with the following error message:

```
Error: Non positive occurence in ((D → D) → D).
```

The definition of `D` should be quite familiar: for instance, in order to build a model for the λ-calculus, one needs a domain `D` that is isomorphic to its own function space `D → D` (each term is also a function), and one could be tempted to simply define it as the smallest type `D` for which there exists an embedding `(D → D) → D`.

Suppose we accept such a definition. We would then expect to be able to reason by case-analysis on an object `d:D`. In particular, `d` must reduce to the form `lam f` for some function `f` and we would expect to be able to extract this information out of `d` by means of the usual pattern matching operation.

This is all we need to write a function that, given two terms in `D`, interprets the first as a function (i.e. unboxes the embedded function) and applies it to the second.

```
definition app : D → D → D :=λd,e. match d with [lam f ⇒ f e].
```

At this point it should be clear that what we are really doing is embedding the λ-calculus into CIC. More importantly, our embedding reuses the binding structures of CIC to express binding in the object calculus. For instance, we can easily express the familiar *duplicator* term of the λ-calculus (i.e. `λx.x x`) as follows:

```
definition dup : D :=lam (λx:D. app x x).
```

However, our embedding goes beyond that: we can even reuse reduction in CIC to express reduction in the object λ-calculus. Since the λ-calculus admits non-terminating terms, making the definition of `D` legal has surreptitiously added non-terminating computations into CIC as well. Let for example:

```
definition omega : D :=app dup dup.
```

Then,

```
omega = app dup dup = (λx:D.app x x) dup = app dup dup = omega
```

hence omega would reduce to itself, resulting in a non-terminating computation!

So, what is so dangerous with non-termination? Consider for instance the following definitions:

```
definition F : D :=lam (λz. lam (λy.z)).    (* λz,y.z *)

definition Fxx : D :=lam (λx.app F (app x x)). (* λx.F (x x) *)

definition YF : D :=app Fxx Fxx.             (* (Fxx Fxx) == Y F *)
```

We used basic results about existence of fixed points in the λ-calculus to build a non-terminating term *YF* that, upon reduction, produces infinite copies of the lam constructor. Then we can define a function size whose purpose is to count the number of constructors used in a term of type *D*:

```
definition I : D :=lam (λx.x).

let rec size d :=match d with [ lam f ⇒ S (size (f I)) ].
```

where the identity term *I* is a dummy, used to obtain from *f* a new term for a recursive call to size. Then, if we try to compute the size of *YF*, we discover that it is equal to `S ((λy.app Fxx Fxx) I) = S (size YF)`, but this is simply absurd, since no natural number can be equal to its successor! The point is that *nat* was supposed to range over natural numbers, while introducing the possibility of divergence, we are surreptitiously extending the domain adding a new "bottom" element.

In order to ensure consistency, the inductive definition must respect some "positivity" conditions concerning the occurrences of the inductive type in the types of the arguments of constructors.

The *polarity* of an occurrence of a propositions *A* inside a propositions *B* is defined in the following way. If *A=B* then *A* occurs positively in *B*. if $B = C \rightarrow D$ or $B = \forall x{:}C.D$, then if *A* occurs positively (negatively) in *D*, then it occurs positively in *B*; conversely, if it occurs positively (negatively) in *C*, then it occurs negatively (positively) in *B*.

For instance *D* occurs two times in $D \rightarrow D$ (the type of the only argument of *embed*); the first occurrence is negative, while the second is positive.

As a first approximation, the strict positivity requirement for inductive definitions imposes that for any constructor

$$c\colon A_1 \rightarrow \ldots \rightarrow A_n \rightarrow X$$

the inductive type *X* can only occur positively in any of the A_i.

To understand the reasons for this restriction we need to make a short digression on recursive domain equations.

When we define a (recursive) inductive type we are essentially defining a type as a solution of a suitable equation. Consider for instance the type of natural numbers. The type of the two constructors can be seen as two injections $O\colon unit \rightarrow nat$ and $S\colon nat \rightarrow nat$; since we have no additional elements, the type of natural numbers satisfies the equation

$$nat \equiv unit \oplus nat$$

The reasoning can be generalized to any inductive type *X*. In particular *X* must be isomorphic to the disjoint sum âŁŢ of the Cartesian product of the types of the constructor's arguments. To take another example, let us consider the type of lists on a parametric type A. In this case:

$$\mathit{list}\ A \equiv \mathit{unit} \oplus A \times \mathit{list}\ A$$

The delicate issue is to ensure the existence of solutions to such recursive domain equations (see e.g. [2], for an introduction). In general, the standard approach to solve a recursive definition is to look for a fixed point. There are a lot of interesting fixed-point theorems in mathematics, but one that is particularly simple and appealing is the Knaster-Tarski theorem. This theorem says that if L is a complete lattice and $f : L \to L$ is a monotone function, then the set of fixed points of f is also a complete lattice. In particular, there is a *least fixed point* μf (as well as a greatest fixed point νf).

A lattice is just a degenerate case of a *category*. Instead of having elements and a partial order relation on them, in a category we have a collection of *objects* and, for each pair of objects a, b a collection $\hom[a, b]$ of *morphisms* from a to b (we write $f : a \to b$ to express that $f \in hom[a, b]$). Morphisms are supposed to be closed under composition, generalizing the transitivity of the order relation, and we are supposed to have identities for any object, generalizing reflexivity.

The notion of monotone function generalizes to the notion of *functor*. A functor F between two categories C and D is a transformation mapping any object a of C into an object $F(a)$ of D, and any morphism $f : a \to b$ into a morphism $F(f) : F(a) \to F(b)$ in D. The functor is supposed to be well-behaved, in the sense that it maps identities to identities ($F(\mathrm{id}_a) = \mathrm{id}_{F(a)}$) and it preserves composition ($F(f \circ g) = F(f) \circ F(g)$).

So, when we try to interpret types defined by a recursive equations of the form $X = F(X)$ an essential prerequisite is to be able to provide a functorial description of the transformation F, and in particular of the action of F on morphisms (i.e. on terms). Consider for instance the equation $X = \mathit{unit} \oplus X$ definining nat. Given a function $f : A \to B$, it is natural to lift it to $F(f) : \mathit{unit} \oplus A \to \mathit{unit} \oplus B$ as $id_{\mathit{unit}} \oplus f$. Now, suppose we have an equation of the form $X = X \to X$; then given $f : A \to B$ we would need to lift f to a morphism in $(A \to A) \to (B \to B)$ and there is no natural way to do it (we miss a transformation from B to A).

The strict positivity condition is a syntactic criterion that guarantees that we can always associate a functorial action with our construction. This is actually directly exploited in the definition of the eliminators for the inductive type.

9.2 Universe Constraints

Another constraint on inductive definitions requires that the sorts of constructors and the sort of the inductive type must be equal. Considering that each constructor for an inductive type T must have a type of the form

$$A_1 \to \ldots \to A_n \to T$$

there are two possibilities: either $T{:}\mathit{Prop}$ and then any argument is accepted (but, as we shall see, with restrictions on the elimination rules), or $T{:}\mathit{Type}[j]$. In this case, for every i such that $A_i{:}\mathit{Type}[k_i]$, we expect to have $k_i \leq j$.

Consider for instance the case of DeqSets:

```
record DeqSet : Type[1] :=
{ carr :>Type[0]
; eqb: carr → carr → bool
; eqb_true: ∀x,y. (eqb x y = true) ↔(x = y)}.
```

As we know, this is syntactic sugar for an inductive type with a single constructor taking as arguments the values of the record fields. So, among the arguments we have the carrier, and since we are quantifying over generic elements in `Type[0]` we shall end up with a larger type, the smallest being `Type[1]`.

In order to understand the reasons for this restriction, we shall prove that its omission essentially opens the way to an encoding of Russell's paradox, as first observed in [23].

Let us consider the following type:

```
inductive U : Type[1] :=c: ∀A:Type[0].(A→ Prop) → U.
```

You may look at `c` as a sort of comprehension axiom that for every property builds the corresponding "set" as an element of the universe `U`. We can define

```
definition carrier :=λu:U.match u with [ c A P ⇒ A].

definition property :=λu:U.
  match u return (λu. carrier u → Prop)
  with [ c A P ⇒ P].
```

so, from each element `u:U` we can extract its characteristic property.

Given two terms `u:U` and `a:A` such that `A=carrier u` we say that `a` is a member of `u` provided `a` satisfies the characteristic property of `u`. A tricky point is that (`property u`) expects as input an element of type `carrier u`, while `a:A`; we know that they are equal, but this does not imply that they are convertible!

What we can do is to use the equality between the two types `A` and `B` to define an *embedding* from `A` to `B`:

```
lemma embed: ∀A,B:Type[0].A=B → B→ A.
#A #B #h >h // qed.
```

The previous embedding has very good properties; in particular, if the two types can be proved to be identical via reflexivity, the corresponding embedding is just the identity.

```
lemma embed_id: ∀A,a.embed A A (refl ...) a = a.
// qed.
```

Using the previous embedding we can now define the following membership relation:

```
definition member : ∀u:U.∀A:Type[0].∀a:A.carrier u = A → Prop :=
  λu,A.λa:A.λh.property u (embed (carrier u) A h a).
```

where in particular

```
lemma member_to_P: ∀A,P,a.
  property (c A P) (embed ?? (refl ??) a) = P a.
```

However, suppose we accept the following definition, with `U:Type[0]`:

```
inductive U : Type[0] :=c: ∀A:Type[0].(A→ Prop) → U.
```

Then, we may look, inside the type `U`, for all those elements having `U` as carrier, and for these elements it would be sound to ask if they belong to themselves or not.

```
definition RussellP :=λu:U.∀h:carrier u = U.¬(member u U u h).
```

Then, we could form the following paradoxical element of `U`:

```
definition Russell : U :=c U RussellP
```

Suppose `(RussellP Russell)`; by expanding the definition of `RussellP`, this is equivalent to

$$\forall h{:}\mathit{carrier}\ \mathit{Russell}{=}U.\ \neg(\mathit{member}\ \mathit{Russell}\ U\ \mathit{Russell}\ h)$$

In particular, we may instantiate `h` with reflexivity, and by `member_to_P` we get

$$\neg(\mathit{RussellP}\ \mathit{Russell})$$

Since `(RussellP Russell)` implies its negation, we conclude `¬(RussellP Russell)`. To get a contradiction, we still need to prove `(RussellP Russell)`, or equivalently,

$$\forall h{:}\mathit{carrier}\ \mathit{Russell}{=}U.\ \neg(\mathit{member}\ \mathit{Russell}\ U\ \mathit{Russell}\ h)$$

If we assume *proof irrelevance* (in fact, Streicher's axiom of the uniqueness of identity proofs is enough) all proofs that `carrier Russell = U` are the same, and we may replace `h` with reflexivity, getting the goal

$$\neg(\mathit{member}\ \mathit{Russell}\ U\ \mathit{Russell}\ h)$$

that we just proved.

9.3 Informative and non informative content

As we observed in the previous section, the following definition is not accepted by Matita since it would be inconsistent:

```
inductive U : Type[0] := c: ∀A:Type[0].(A→ Prop) → U.
```

However, this is prefectly legal:

```
inductive U : Prop := c: ∀A:Prop.(A→ Prop) → U.
```

and one could (and should) naturally wonder how we avoid the paradox in this case. What happens is that the definitions of `carrier` and `property` are rejected. For instance, if we enter the following definition

```
definition carrier :=λu:U.match u with [c A P ⇒ A].
```

we would now get the error message

```
Sort elimination not allowed: Prop towards Type.
```

The intuition is that proofs (elements of sort `Prop`) should be merely understood as certificates testifying that their type is inhabited, but are not supposed to carry any additional information. This can be further stressed by the principle of *proof irrelevance*, asserting that all proofs of a same proposition *A* are *provably equal* inside the logic. In this sense, proofs are *non informative*: no kind of inspection can be perfomed on proofs, and no data can be extracted from them. For instance, the previous definition of *carrier* is trying to extract from the proof *u*:*U*:*Prop* the carrier, that is an observable information: some propositions could have carrier *True*, and some others carrier *False*, and *True* is provably different from *False*. So we would be able to discriminate proofs according to the extracted carrier, contradicting the principle of proof irrelevance.

To take a simple example, suppose we have a proof of *A*∨ *B*; we know that this is either a proof of *A* or a proof of *B* and we could be tempted to define a function distinguishing among these two situations, returing *True* in the first case and *False* in the second.

```
definition discriminate_or :=λA,B.λp:A∨ B.
  match p with
  [or_intror _  ⇒ True
  |or_introl _  ⇒ False ].
```

Suppose we accept the previous definition, and consider the following proofs of *True*∨ *True* that we can assume to be equal by proof irrelevance:

```
axiom eq_proofs: or_introl True True I = or_intror True True I.
```

But then, *True* would be equal to *discriminate_or* (*or_intro_l or_True True I*) that is equal to *discriminate_or* (*or_intro_r or_True True I*) that is equal to *False*!

To make another example, let us see how we can derive the paradox of the previous section axiomatizing the existence of *carrier* and *property*.

```
inductive U : Prop :=c: ∀A:Prop.(A→ Prop) → U.

axiom carrier : U → Prop.
axiom carrier_def : ∀A,P. carrier (c A P) = A.

axiom property : ∀u:U. carrier u → Prop.
axiom property_def: ∀A,P,a.
  property (c A P) (embed ?? (carrier_def A P) a) = P a.

definition member : ∀u:U.∀A:Prop.∀a:A.carrier u = A → Prop :=
  λu,A.λa:A.λh.property u (embed (carrier u) A h a).

lemma member_to_P: ∀A,P,a.member (c A P) ? a (carrier_def A P) = P a.
#A #P #a whd in match (member ????); // qed.

definition RussellP :=λu:U.∀h:carrier u = U.¬ (member u U u h).
definition Russell :=c U RussellP.

lemma not_RussellP_Russell: RussellP Russell → False.
whd in match Russell; whd in ⊢(%→ ?); #H
cases (H (carrier_def ??)) #H1 @H1 >member_to_P @H
```

```
qed.
```

Moreover, assuming proof irrelevance

```
axiom irrel: ∀A,P.∀h1,h2: carrier (c A P) = A. h1 = h2.

lemma RussellP_Russell: RussellP Russell.
whd in match Russell; #h % cut (carrier_def ?? = h) [@irrel]
#Hcut <Hcut >member_to_P @not_RussellP_Russell
qed.
```

10. FURTHER READINGS

Unfortunately, there is no really good theoretical introduction to modern, dependent type theory. The best textbook probably remains old Martin-Löf's monograph [16] that, however, does not address any major metatheoretical issue (normalization, consistency, etc.).

For the simply typed lambda calculus, system T, and system F (i.e. the polymorphic lambda calculus) an excellent introductory text is the book by Girard et al. [14]; in it, the reader may also find a clear explanation of the so called impredicative encoding of inductive types, that is a way for expressing inductive types using higher order polymorphism.

For pure type systems (PTSs), i.e. various systems of typed lambda calculi comprising types dependent on terms but *not* explicit inductive definitions, a very clear (but a bit dry) reference is Barendergt's monograph [12]. Alternatively, you may consult the conspicuous book by Sorensen and Urzyczyn on the Curry-Howard analogy [21].

Concrete models of the Calculus of Inductive Construction have been mostly investigated by Miquel et al. [17]. For a categorical investigation, and the relation between type theory and categorical logic, see Jacobs's book [15].

Recently, a very interesting relation between Type Theory and Omotopy Theory has been discovered, giving birth to Homotopy Type Theory (HoTT). People interested in this topic can start from [22].

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